# Placement Exam (Foundations)
# Solutions

CyberMath Academy

Summer 2022

## Section 1: Multiple Choice

1. What is the value of $1+2+3+4+\cdots+25$?

   (a) 250
   (b) 275
   (c) 300
   (d) 325
   (e) 350

   Answer: $\boxed{D}$

   Solution: An important fact to remember in general is the equation

   $$1+2+3+\cdots+n=\frac{n(n+1)}{2}.$$

   This is Gauss' famous method of summing the first $n$ positive integers, and works because we are "pairing together" the first and last terms, then the second and second-to-last terms, and so forth. We get $\frac{25\cdot 26}{2}=25\cdot 13=\boxed{325}$ as our sum, in this case, with $n=25$.

2. How many seventeens are in ten ones plus three fifteens plus four sixteens?

   (a) 5
   (b) 6
   (c) 7
   (d) 8
   (e) 9

   Answer: $\boxed{C}$

   Solution: This is $\frac{10\cdot 1+3\cdot 15+4\cdot 16}{17}=\frac{119}{17}=\boxed{7}$.

3. What is the largest positive integer $n$ for which half of $n$ plus one is at least two-thirds of $n$?

   (a) 4
   (b) 6
   (c) 7
   (d) 8
   (e) 9

Answer: $\boxed{B}$

Solution: We have $\frac{1}{2}n + 1 \geq \frac{2}{3}n$, so $1 \geq \frac{1}{6}n$, upon subtracting $\frac{1}{2}n$ from both sides of the inequality. Then, multiplying through by 6, we have $\boxed{6} \geq n$.

4. How many $2 \times 4$ squares can fit inside a $12 \times 16$ rectangle?

   (a) 24
   (b) 32
   (c) 36
   (d) 48
   (e) 72

   Answer: $\boxed{A}$

   Solution: Because 12 is divisible by 2 and 16 is divisible by 4 (and this would work in either order; for good measure, 12 is divisible by 4 and 16 is divisible by 2 as well), this is $\frac{12 \cdot 16}{2 \cdot 4} = \boxed{24}$.

5. Today is Tuesday. What day of the week will it be 60 days from today?

   (a) Thursday
   (b) Friday
   (c) Saturday
   (d) Sunday
   (e) Monday

   Answer: $\boxed{C}$

   Solution: The remainder when 60 is divided by 7 is 4, and the day 4 days after Tuesday is $\boxed{\text{Saturday}}$.

6. I have a bunch of apples. If I sort them into groups of five, I will have two left over, but if I sort them into groups of eight, I will have three left over. What is the fewest number of apples I may have?

   (a) 12
   (b) 27
   (c) 42
   (d) 57
   (e) 77

   Answer: $\boxed{B}$

   Solution:

7. What is the sum of all numbers that are twice as far away from 24 as from 48?

   (a) 40
   (b) 72
   (c) 96
   (d) 112
   (e) 136

Answer: $\boxed{D}$

Solution: If $x$ is twice as far away from 24 as from 48, and $x > 48$, then $x - 24 = 2(x - 48)$, or $x - 24 = 2x - 96$, and $x = 72$. If $x < 24$ instead, then $24 - x = 2(48 - x)$, or $x = 72$, but this is a contradiction. Finally, if $24 < x < 48$, then $x - 24 = 2(48 - x) = 96 - 2x$, so that $x = 40$. Hence, the requested sum is $72 + 40 = \boxed{112}$.

8. A rectangle has integer side lengths and diagonal length $3\sqrt{10}$. What is its perimeter?

   (a) 16
   (b) 18
   (c) 20
   (d) 22
   (e) 24

   Answer: $\boxed{E}$

   Solution: By the Pythagorean Theorem, the sum of the squares of the side lengths is $(3\sqrt{10})^2 = 90$. One-third of the side lengths must also have a sum of squares of 10, but the only integers whose squares sum to 10 are 1 and 3, so the side lengths must be 3 and 9. Indeed, $3^2 + 9^2 = 90$, so the perimeter is $2(3 + 9) = \boxed{24}$.

9. How many different ways are there to arrange the letters in the word SUMMER?

   (a) 180
   (b) 240
   (c) 360
   (d) 480
   (e) 720

   Answer: $\boxed{C}$

   Solution: This is $\frac{6!}{2!} = \boxed{360}$, because there are $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ ways to permute a string of 6 different letters, but we divide by $2!$ to account for the fact that the order of the two Ms does not matter.

10. What is the value of $\frac{27^4}{243^2}$?

    (a) 1
    (b) 3
    (c) 9
    (d) 27
    (e) 81

    Answer: $\boxed{C}$

    Solution: This is $\frac{(3^3)^4}{(3^5)^2} = \frac{3^{3\cdot4}}{3^{5\cdot2}} = \frac{3^{12}}{3^{10}} = 3^{12-10} = 3^2 = \boxed{9}$.

11. A *triangular number* is a positive integer that can be written as $1 + 2 + 3 + \cdots + n$ for some positive integer $n$. How many three-digit triangular numbers are there?

    (a) 31
    (b) 32

(c) 35
(d) 40
(e) 45

Answer: $\boxed{A}$

Solution: Recall that $1+2+3+\cdots+n = \frac{n(n-1)}{2}$. The smallest three-digit triangular number is $1+2+3+\cdots+14 = 15\cdot 7 = 105$, and the largest comes from the largest $n$ such that $n(n+1) \leq 1998$. Since $45\cdot 44 = 1980$, this $n$ is 44, and so there are $44-14+1 = \boxed{31}$ three-digit triangular numbers.

12. A regular polygon has at least 100 interior diagonals. What is the fewest number of sides it may have?

(a) 10
(b) 11
(c) 13
(d) 15
(e) 16

Answer: $\boxed{E}$

Solution: The number of diagonals in a regular $n$-gon is $\frac{n(n-3)}{2}$ (because, given a choice of vertex, all but 3 vertices are not adjacent to that vertex, and hence can be the ending point of a diagonal). We want $n(n-3) \geq 200$, so $n \geq \boxed{16}$.

13. The sum of three positive integer multiples of 3 is a positive integer ending in 5. What is the smallest possible product of the three integers?

(a) 72
(b) 81
(c) 90
(d) 108
(e) 120

Answer: $\boxed{B}$

Solution: The sum of three multiples of 3 is itself a multiple of 3. The smallest multiple of 3 ending in 5 is 15, so the integers can be 3, 3, and 9, or 3, 6, and 6, in some order. The smaller of the two resulting products is $\boxed{81}$.

14. An even positive integer less than or equal to 10, and a positive integer multiple of 3 less than or equal to 10, are selected at random. What is the probability that their sum is a multiple of 6?

(a) $\frac{1}{15}$
(b) $\frac{2}{15}$
(c) $\frac{1}{5}$
(d) $\frac{4}{15}$
(e) $\frac{1}{3}$

Answer: $\boxed{A}$

Solution: Note that 6 is not a possible sum, and neither is 18. The only way in which 12 can be expressed as a sum of these two integers is as $6+6$, so the probability is $\frac{1}{5\cdot 3} = \boxed{\frac{1}{15}}$.

15. If we multiply 2 by itself $n$ times, and then divide the result by 105, the remainder will be 1. What is the smallest possible value of $n$?

   (a) 9

   (b) 10

   (c) 12

   (d) 15

   (e) 16

   Answer: $\boxed{C}$

   Solution: By the Chinese remainder theorem, we can consider $2^n$ modulo 7 and 15. Notice that $7 = 2^3 - 1$ and $15 = 2^4 - 1$, so $2^3 \equiv 1 \bmod 7$ and $2^4 \equiv 1 \bmod 15$. Then the $n$ must be a multiple of both 3 and 4, so it can be as small as $\boxed{12}$.

## Section 2: Short Answer

16. What is the total number of digits in all of the question numbers (1-24) on this test?

    Answer: $\boxed{39}$

    Solution: The numbers 1-9 have 9 digits in total, and the 15 two-digit numbers from 10-24 have 30 digits in total, giving $\boxed{39}$ digits in all.

17. How many positive integers have a square root that is at least 4, but less than 5?

    Answer: $\boxed{9}$

    Solution: These integers are 16 through 24, since $4^2 = 16$ and $5^2 = 25$; there are $24 - 16 + 1 = \boxed{9}$ of these.

18. How many three-digit positive integers are perfect squares?

    Answer: $\boxed{22}$

    Solution: The smallest three-digit perfect square is $10^2 = 100$ and the largest is $31^2 = 961$, so we have $31 - 10 + 1 = \boxed{22}$ three-digit squares.

19. A rectangle with integer side lengths, in units, has perimeter 22 units. What is its largest possible area, in square units?

    Answer: $\boxed{30}$

    Solution: We want the side lengths to be as close to each other as possible; they sum to 11, so they should be 5 and 6, and their product is $\boxed{30}$.

20. What is the volume, in cubic units, of a cube whose edge lengths sum to 60 units?

    Answer: $\boxed{125}$

    Solution: A cube has 12 congruent edges, so each edge has length 5. The volume of a cube with edge length $s$ is $s^3$, so the volume is $5^3 = \boxed{125}$.

21. What is the exponent of the largest power of 3 that is a factor of $22! = 22 \cdot 21 \cdot 20 \cdots 1$?

    Answer: $\boxed{9}$

    Solution: We have 7 multiplies of 3, 2 multiples of 9, and no multiples of 27 or any larger power of 3. This gives us a factor of $3^9$ but not $3^{10}$, and so the exponent is $\boxed{9}$.

22. What is the sum of all the possible remainders when a number of the form $2^k$, where $k$ is a non-negative integer, is divided by 20?

    Answer: $\boxed{43}$

    Solution: Starting from 1 and multiplying by 2 repeatedly, the possible remainders are 1, 2, 4, 8, 16, and 12, after which we get 4 again and the cycle repeats. The sum of these is $\boxed{43}$.

23. What is the value of

$$1 \cdot 2 + 3 \cdot 4 + 5 \cdot 6 + 7 \cdot 8 + \cdots + 99 \cdot 100?$$

Answer: $\boxed{169150}$

Solution: This is the sum of $(2k-1)(2k)$ from $k=1$ to $k=50$, which is the sum of $4k^2-2k$ from $k=1$ to $k=50$. Because we have

$$1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$$

and

$$1+2+3+\cdots+n=\frac{n(n+1)}{2}, \tag{1}$$

the sum is $4\cdot\frac{50\cdot51\cdot101}{6}-2\cdot\frac{50\cdot51}{2}=100\cdot17\cdot101-50\cdot51=10100\cdot17-150\cdot17=9950\cdot17=\boxed{169150}$.