Placement Exam (Mastery Level)

CyberMath Academy

Summer 2022

Instructions (please read!)

Solve as many of the following 50 problems (worth 150 points) as you can in 2 hours. The exam will be split into three sections. Section 1 is the Multiple Choice section, section 2 is the Short Answer section, and section 3 is the Free Response section. The point value of each question will be indicated next to the problem.

In Sections 1 and 2, if you obtain the correct answer, you will receive full credit, regardless of whether or not you show your work. On the other hand, even if your answer is incorrect, you still have the chance to earn partial credit if you provide some justification for your answer. **In section 3, the Free Response section, please be sure to fully justify all of your answers, showing all work!** Correct answers without justification will not receive any points. Incorrect answers with justification may still be eligible for partial credit.

In general, the questions are increasing in difficulty within each section; question 1 is intended to be fairly approachable, whereas the last question is meant to be difficult. Don't let this discourage you from solving questions out of order, though; choose whichever problems you like, and strategize according to whichever you feel that you are most likely to solve within the time limit.

Please also note that this exam has no set "passing" or "failing" score. In particular: *you do not need anywhere near the "traditional" score of 70 percent to pass!!* In fact, if you score 50 percent of the available points, you are likely very well qualified for the Mastery section of the High School Math course. This is an exam meant to see where you top out and to expose you to new ideas, so don't worry if you haven't seen everything before – after all, that's what the camps are for!

Best of luck!! :)

Section 1: Multiple Choice

1. What is the value of

$$|1-|2-|3-|4-|5-|6-7||||||?$$

   (a) -2
   (b) -1
   (c) 0
   (d) 1
   (e) 2

2. Anna and Ben are competing in a race. Anna finishes in 7 minutes and 5 seconds, which is 1 minute and 25 seconds faster than Ben. What percent faster is Anna than Ben?

   (a) $\frac{50}{3}$
   (b) 20
   (c) $\frac{70}{3}$
   (d) 22.5
   (e) 25

3. What is the smallest perfect square larger than 1 that is also a perfect cube?

   (a) 27
   (b) 36
   (c) 49
   (d) 64
   (e) 81

4. A fair coin is flipped four times in a row. What is the probability that it comes up heads at least half of the time? Express your answer as a common fraction.

   (a) $\frac{1}{2}$
   (b) $\frac{9}{16}$
   (c) $\frac{5}{8}$
   (d) $\frac{11}{16}$
   (e) $\frac{3}{4}$

5. What is the positive difference between the solutions x to the equation

$$x^2-5x+1=0?$$

   Express your answer in simplest radical form.

   (a) $\sqrt{6}$
   (b) $\sqrt{21}$
   (c) $2\sqrt{6}$
   (d) $\sqrt{29}$
   (e) $2\sqrt{13}$

6. The base-10 positive integer N has a value of 106 in base 7. What is the value of N in base 13?
   - (a) 39
   - (b) 43
   - (c) 4A
   - (d) 57
   - (e) 70
7. How many three-digit positive integers contain at least two prime digits?
   - (a) 256
   - (b) 320
   - (c) 336
   - (d) 360
   - (e) 424
8. How many non-congruent acute triangles have all angle measures equal to a positive integer number of degrees?
   - (a) 585
   - (b) 630
   - (c) 660
   - (d) 675
   - (e) 720
9. In how many ways can I roll four fair six-sided dice, each labeled from 1-6, so that I end up with rolls whose values are in arithmetic progression, in some order?
   - (a) 30
   - (b) 51
   - (c) 63
   - (d) 75
   - (e) 78
10. Triangle ABC has side lengths $AB = 3$, $BC = 4$, and $CA = 5$. Triangle DEF is similar to ABC and has perimeter, in units, numerically equal to 10 percent of its area, in square units. What is its area, in square units?
    - (a) 1000
    - (b) 1200
    - (c) 1800
    - (d) 2400
    - (e) 2500
11. How many ordered triples (x, y, z) of positive integers have a product of 2022?
    - (a) 25
    - (b) 27
    - (c) 30
    - (d) 36
    - (e) 40

12. Triangle ABC has side lengths $AB = 6$, $BC = 8$, and $CA = 10$. A circle with center O is inscribed in $\triangle ABC$. What is the square of the length of OC?

    (a) 20
    (b) 28
    (c) 32
    (d) 38
    (e) 40

13. In how many ways can 16 be written as a sum of prime positive integers in non-increasing order? (For example, $7 + 5 + 2 + 2$ is one such representation.)

    (a) 11
    (b) 12
    (c) 14
    (d) 16
    (e) 17

14. If n is a positive integer for which $\sqrt{n+1} - \sqrt{n} < 0.1$, what is the smallest possible value of n?

    (a) 24
    (b) 25
    (c) 26
    (d) 49
    (e) 50

15. Albert and George are playing a very large number of games of rock-paper-scissors. So far, Albert has won 50.5 percent of the games, and George has won the remaining 49.5 percent. If George were to somehow win the next 400 games in a row (!), Albert will have won 49.5 percent of all the games played up to that point, while George will retake the lead with 50.5 percent of the games won. How many games have they played so far, not including the 400 games that would put George in the lead?

    (a) 19200
    (b) 19600
    (c) 19800
    (d) 20000
    (e) 20200

16. A magical coin has the property that all flips until the point when at least one heads and at least one tails have been flipped have probability $\frac{1}{2}$ of coming up heads and probability $\frac{1}{2}$ of coming up tails. After both heads and tails have come up at least once, the probability of the coin coming up heads is $\frac{h}{h+t}$ and the probability of it coming up tails is $\frac{t}{h+t}$, where h and t are the numbers of heads and tails flipped up to that point, respectively. What is the probability that, out of 5 flips, exactly 4 are heads?

    (a) $\frac{5}{32}$
    (b) $\frac{3}{16}$
    (c) $\frac{7}{32}$
    (d) $\frac{17}{64}$
    (e) $\frac{9}{32}$

17. Let r and s be the roots of $x^2 + 20x + 22$, and let t and u be the roots of $x^2 + 20x + 23$. What is the value of $rt + ru + st + su$?

    (a) 200
    (b) 305
    (c) 400
    (d) 485
    (e) 548

18. The number 2023 has the property that all of its nonzero digits have a product of 12. How many four-digit positive integers have this property?

    (a) 63
    (b) 66
    (c) 72
    (d) 78
    (e) 81

19. What is the sum of the base-16 logarithms of all of the positive integer divisors of 4096? Express your answer as a common fraction.

    (a) $\frac{55}{4}$
    (b) $\frac{33}{2}$
    (c) $\frac{39}{2}$
    (d) $\frac{91}{4}$
    (e) $\frac{105}{4}$

20. For every real number x, let $\lfloor x \rfloor$ be the greatest integer less than or equal to x. For example, $\lfloor 5.99 \rfloor = 5$ and $\lfloor -\pi \rfloor = -4$. What is the smallest positive integer n such that

$$\lfloor \sqrt{1} \rfloor + \lfloor \sqrt{2} \rfloor + \lfloor \sqrt{3} \rfloor + \cdots + \lfloor \sqrt{n} \rfloor \geq 100?$$

    (a) 30
    (b) 37
    (c) 44
    (d) 51
    (e) 58

21. Rectangle $ABCD$ has positive integer side lengths r and s. Points E and F trisect $\overline{AB}$, points G and H trisect $\overline{BC}$, points I and J trisect $\overline{CD}$, and points K and L trisect $\overline{DA}$. If the area of octagon $EFGHIJKL$ is 2023, compute the number of possible ordered pairs (r, s).

    (a) 8
    (b) 9
    (c) 10
    (d) 12
    (e) 15

22. How many ordered tuples of positive integers have the property that the sum of the squares of the integers is 10?

    (a) 15
    (b) 16
    (c) 18
    (d) 20
    (e) 21

23. Rectangle $ABCD$ has side lengths $AB = 4$ and $BC = 5$. Point E is the midpoint of $\overline{AB}$, and point F lies on $\overline{BC}$ such that $BF = 4$ and $FC = 1$. Line segments $\overline{AF}$ and $\overline{DE}$ intersect at point G. What is $\frac{AG}{AF}$? Express your answer as a common fraction.

    (a) $\frac{1}{3}$
    (b) $\frac{5}{14}$
    (c) $\frac{2}{5}$
    (d) $\frac{7}{16}$
    (e) $\frac{9}{20}$

24. For each positive integer n, let $Z(n)$ be the number of 0s in which n ends. For example, $Z(20220000) = 4$ and $Z(10001) = 0$. What is the sum of $Z(n)$ over all positive integer factors of 10^{10}?

    (a) 120
    (b) 165
    (c) 220
    (d) 286
    (e) 325

25. How many ordered triples of positive integers have a least common multiple of exactly 6?

    (a) 42
    (b) 45
    (c) 47
    (d) 49
    (e) 50

26. A geometric progression has a non-integer (but rational) common ratio, and has 7 terms, all of which are all integers. If not every term is a multiple of the last term, what is the smallest possible sum of all terms in the geometric progression?

    (a) 254
    (b) 665
    (c) 1093
    (d) 2059
    (e) 2186

27. How many permutations of $(1, 2, 3, 4, 5)$ have three consecutive elements a, b, and c, in that order, for which $a + b = c$?

(a) 16
(b) 20
(c) 22
(d) 28
(e) 34

28. Two complex numbers x and y satisfy the equations

$$x^2 + y^2 = 3,$$
$$x^2 + 2xy - y^2 = 7.$$

Given that it is positive, what is the imaginary part of y^2? Express your answer as a common fraction in simplest radical form.

(a) $\frac{\sqrt{17}}{4}$
(b) $\frac{3\sqrt{3}}{4}$
(c) $\frac{\sqrt{31}}{4}$
(d) $\frac{\sqrt{33}}{4}$
(e) $\frac{\sqrt{65}}{4}$

29. Let x and y be positive real numbers chosen uniformly at random from the set of ordered pairs (x, y) of positive real numbers such that $x + y = 3$. What is the expected value of the greatest integer not exceeding xy? Express your answer as a common fraction in simplest radical form.

(a) $\frac{\sqrt{3}+1}{3}$
(b) $\frac{\sqrt{5}+1}{3}$
(c) $\frac{2\sqrt{7}-3}{2}$
(d) $\frac{\sqrt{5}+3}{3}$
(e) $\frac{2\sqrt{13}-1}{2}$

30. Let triangle ABC have side lengths $AB = 7$, $BC = 9$, and $CA = 8$. Point D lies on $\overline{BC}$ with $\tan(m\angle BAD) = 2\tan(m\angle DAC)$. The value of $\tan(m\angle DAC)$ can be written in the form $\frac{\sqrt{p}-\sqrt{q}}{r}$, where p, q, and r are positive integers and neither p nor q is a multiple of the square of a prime number. What is $p + q + r$?

(a) 18
(b) 21
(c) 32
(d) 62
(e) 70

Section 2: Short Answer (40 pts.)

Suggested time: 30 minutes.

Each question in this section ONLY requires a number as the final answer. If you would like, you can show your work to try and receive partial credit in case your answer is incorrect. Each correct answer in this section is worth 2.5 points.

31. A positive integer with two digits is a multiple of 3, and its digits sum to a prime number. What is the sum of the possible values of this integer?

32. How many integers from 0 through 7, inclusive, could be the remainder when the sum of two (not necessarily different) positive integer perfect squares is divided by 8?

33. For how many positive integers n does 2022 leave a remainder of 2 when divided by n?

34. Let $d(n)$ be the units digit of n. What is the value of
$$d(1^2) + d(2^2) + d(3^2) + \cdots + d(2022^2)?$$

35. A fair six-sided die is labeled with the positive integers from 1 through 6, inclusive. What is the expected value of the square of a single roll? Express your answer as a common fraction.

36. Rectangle $ABCD$ has side lengths $AB = 4$ and $BC = 3$. Point E lies in the plane of $ABCD$, on the same side of $\overline{AC}$ as point B, so that $\overline{AE} \perp \overline{AC}$ and $AE = 10$. What is the length of line segment $\overline{DE}$? Express your answer in simplest radical form.

37. In how many ways can we choose two distinct positive integers from the set $\{1, 2, 3, \cdots, 100\}$ that sum to a multiple of 3?

38. If x and y are positive real numbers such that $x^2 + y^2 = 23$ and $x^3 + y^3 = 110$, what is the value of $x^4 + y^4$?

39. Two distinct integers from 1 through 10, inclusive, are chosen uniformly at random. What is the expected value of their greatest common factor? Express your answer as a common fraction.

40. For some positive integer n, the triangle with side lengths 2, $\sqrt{17}$, and $\sqrt{n}$ has area 1. What is the sum of all possible values of n?

41. (Source: BmMT 2021) What is the sum of $\frac{1}{n}$ over all positive integer factors n of 360? Express your answer as a common fraction.

42. How many ordered tuples of positive integers in non-decreasing order have a sum no larger than 10?

43. Triangle ABC has $AB = 13$, $BC = 14$, and $CA = 15$. Points D and E lie on $\overline{AB}$ and $\overline{CD}$, respectively, such that $m\angle DAE = m\angle EAC$. If $DE = 2$, what is the largest integer less than or equal to 10 times the length of $\overline{AD}$?

    The following questions are part of a *cyclic relay,* in which question 44 uses the answer to question 45, question 45 uses the answer to question 46, and question 46 uses the answer to question 44. All answers are positive integers.

44. Let B be the answer to question 45. In how many ways (with respect to order) can we construct a list of three elements chosen from either B or C, such that their sum is an odd multiple of 3? ($(2, 2, 5)$ and $(5, 2, 2)$ are two different lists, for example.)

45. Let C be the answer to question 46. What is the geometric mean of A and C?

46. Let A be the answer to question 44. What is the largest positive integer k for which 2^k is a factor of A^3B^3?

Section 3: Free Response (80 pts.)

Suggested time: 1 hour.

Each question in this section requires a FULL ESSAY RESPONSE. If you provide just a number as your answer, you will receive zero points on that question, even if your answer is correct.

Many of the questions are primarily definitional and/or conceptual; others are meant to be more similar to questions you might see in an actual proof-based competition.

47. **[35]** Definitions and their applications.

    (a) **[3]** Please state the Binomial Theorem, and use it to find the coefficient of x^5y^2 in

    $$(3x+2y)^7.$$

    (b) **[3]** What is the Pigeonhole Principle? Please use it to show that, for any three positive integers, the sum of some two of them must always be even.

    (c) **[4]** Please explain the combinatorial concept of *stars and bars,* and apply it to solve a problem of your choosing.

    (d) **[5]** Please state Vieta's formulas for a quadratic polynomial. How can we calculate the sum of the squares of the roots of a quadratic polynomial? What about the sum of the fourth powers? Can you generalize this to polynomials of arbitrary degree?

    (e) **[6]** Please carefully state de Moivre's theorem. Why can we represent complex numbers in terms of sine and cosine? How do you think we might define sine and cosine for complex arguments?

    (f) **[7]** Please define the *power* of a point with respect to a circle, and carefully state the *power of a point* theorem in all of its forms. Please sketch a proof of at least one of those forms.

    (g) **[7]** What is Euler's phi/totient function, and why is it a multiplicative function? Please carefully state and prove Fermat's little theorem and Euler's totient theorem, and explain why FLT is a direct consequence of Euler's theorem.

48. **[25]** Conceptual questions.

    (a) **[6]** Explain why the following formulas for the area of a triangle are valid: $\frac{1}{2}bh$, $\frac{1}{2}ab\sin C$, Heron's formula, Shoelace formula.

    (b) **[4]** Explain why the centroid cuts each median of a triangle into a 2:1 ratio.

    (c) **[5]** Can you prove the formulas for sine, cosine, and tangent addition/subtraction?

    (d) **[10]** Building upon the previous part, the *Chebyshev polynomials of the first kind* are defined by

    $$T_n(\cos(\theta)) := \cos(n\theta), \tag{1}$$

    and the *Chebyshev polynomials of the second kind* are defined by

    $$U_n(\cos(\theta)) := \frac{\sin((n+1)\theta)}{\sin(\theta)} \tag{2}$$

    for all non-negative integers n and real numbers θ.

    i. **[7]** Explain why we have linear recurrence relations for T_n and for U_n, and write explicitly what the recurrence relations are. Why do you think they're so similar? What does all of this have to do with de Moivre's formula?

    ii. **[3]** The Chebyshev polynomials of the first kind are useful in approximation theory as the polynomials (with the largest possible leading coefficient) whose absolute value on $[-1, 1]$ is at most 1. Because of this, their roots, which are called *Chebyshev nodes,* are used in polynomial interpolation of continuous functions; the resulting interpolant is a close approximation of the function. Intuitively speaking, where do you think this connection comes from?

49. **[20]** Applied contest-like questions.

(a) **[4]** Compute the remainder when

$$\underbrace{202220222022\cdots2022}_{\text{2022 copies of 2022}}$$

is divided by 33.

(b) **[6]** (Source: SMT 2019) Let n be a real number. What is the maximum possible value of

$$|\sqrt{n^2+4n+5}-\sqrt{n^2+2n+5}|?$$

(c) **[10]** Let r, s, and t be the complex roots of

$$x^3-x^2+x+1.$$

Compute

$$\frac{1}{r^3}+\frac{1}{s^3}+\frac{1}{t^3}.$$

50. **[?]** And now for something completely different...

Now's your chance to be the teacher: come up with your own contest question and solve it in detail. The more difficult your question, the more points you will receive, if you solve it correctly. Noble attempts on harder problems will be rewarded with partial credit.