Placement Exam (Mastery)
Solutions

CyberMath Academy

Summer 2022

Section 1: Multiple Choice

1. What is the value of

$$|1-|2-|3-|4-|5-|6-7||||||?$$

   (a) -2
   (b) -1
   (c) 0
   (d) 1
   (e) 2

   Answer: $\boxed{C}$

   Solution: We compute from right to left: $|6-7|=1$, $|5-1|=4$, $|4-4|=0$, $|3-0|=3$, $|2-3|=1$, and $|1-1|=\boxed{0}$.

2. Anna and Ben are competing in a race. Anna finishes in 7 minutes and 5 seconds, which is 1 minute and 25 seconds faster than Ben. What percent slower is Ben than Anna?

   (a) 20
   (b) 22.5
   (c) 25
   (d) 27.5
   (e) 30

   Answer: $\boxed{A}$

   Solution: Anna's time is $7\cdot 60+5=425$ seconds, while Ben's time is $425+60+25=510$ seconds. Ben is 85 seconds slower than Anna, or $\frac{8500}{425}\%=\boxed{20\%}$ slower.

3. What is the smallest perfect square larger than 1 that is also a perfect cube?

   (a) 27
   (b) 36
   (c) 49
   (d) 64
   (e) 81

Answer: $\boxed{D}$

Solution: The smallest such integer is the smallest perfect sixth power greater than 1, since 6 is the least common multipe of 2 and 3. This is $2^6 = \boxed{64}$.

4. A fair coin is flipped four times in a row. What is the probability that it comes up heads at least half of the time? Express your answer as a common fraction.

   (a) $\frac{1}{2}$
   (b) $\frac{9}{16}$
   (c) $\frac{5}{8}$
   (d) $\frac{11}{16}$
   (e) $\frac{3}{4}$

   Answer: $\boxed{D}$

   Solution: The number of ways for the coin to come up heads $0 \leq h \leq 4$ times out of four is $\binom{4}{h}$. The total number of outcomes for the coin flips is $2^4 = 16$, giving a probability of $\frac{\binom{4}{2}+\binom{4}{3}+\binom{4}{4}}{16} = \boxed{\frac{11}{16}}$.

5. What is the positive difference between the solutions x to the equation

$$x^2 - 5x + 1 = 0?$$

   Express your answer in simplest radical form.

   (a) $\sqrt{6}$
   (b) $\sqrt{21}$
   (c) $2\sqrt{6}$
   (d) $\sqrt{29}$
   (e) $2\sqrt{13}$

   Answer: $\boxed{B}$

   Solution: By the quadratic formula, the solutions are $x = \frac{5\pm\sqrt{21}}{2}$, so the difference is $\boxed{\sqrt{21}}$.

6. The base-10 positive integer N has a value of 106 in base 7. What is the value of N in base 13?

   (a) 39
   (b) 43
   (c) 4A
   (d) 57
   (e) 70

   Answer: $\boxed{B}$

   Solution: We have $N = 1 \cdot 7^2 + 0 \cdot 7^1 + 6 \cdot 7^0 = 55$ in base 10, which is $\boxed{43}$ in base 13, since $55 = 4 \cdot 13^1 + 3 \cdot 13^0$.

7. How many three-digit positive integers contain at least two prime digits?

   (a) 256
   (b) 320

(c) 336
(d) 360
(e) 424

Answer: $\boxed{C}$

Solution: The prime digits are 2, 3, 5, and 7. If the hundreds digit and one other digit are prime, we have $4^2 = 16$ choices for their values, and 6 choices for the value of the other digit. On the other hand, if the tens and units digits are prime, we only have 5 choices for the hundreds digit (since it cannot be zero), giving 80 additional numbers. If all three digits are prime, we get an additional $4^3 = 64$ numbers, for a total of $\boxed{336}$ numbers.

8. How many non-congruent acute triangles have all angle measures equal to a positive integer number of degrees?

   (a) 585
   (b) 630
   (c) 660
   (d) 675
   (e) 720

   Answer: $\boxed{D}$

   Solution: If the smallest angle measures 1°, the sum of the other two angles is 179°, so we have no possible triangles. If the smallest angle measure is 2°, we have 1 possible triangle (2-89-89). If the smallest is 3°, we have 1 possible triangle, if the smallest is 4°, then we have 2 triangles, if the smallest is 5°, 2 triangles, and so forth until the smallest is 45°, at which point we get 22 triangles (45-46-89 through 45-67-68). With a smallest angle of 46°, we have 22 triangles, with a smallest of 47°, 20 triangles, with a smallest of 48°, 19 triangles, then 17 and 16 triangles for a smallest measure of 49° and 50°, respectively, up until 60° which gives a single triangle. Altogether, this yields $(1{+}1{+}2{+}2{+}\cdots{+}22{+}22){+}(22{+}20{+}19{+}17{+}16{+}14{+}\cdots{+}2{+}1) = 506{+}(42{+}36{+}30{+}\cdots{+}6){+}1 = \boxed{675}$ triangles.

9. In how many ways can I roll four fair six-sided dice, each labeled from 1-6, so that I end up with rolls whose values are in arithmetic progression, in some order?

   (a) 30
   (b) 51
   (c) 63
   (d) 75
   (e) 78

   Answer: $\boxed{E}$

   Solution: The possible arithmetic progressions are the 6 constant progressions, and the permutations of 1234, 2345, and 3456, since a common difference of at least 2 is not possible. There are $6 + 4! \cdot 3 = \boxed{78}$ such permutations.

10. Triangle ABC has side lengths $AB = 3$, $BC = 4$, and $CA = 5$. Triangle DEF is similar to ABC and has perimeter, in units, numerically equal to 10 percent of its area, in square units. What is its area, in square units?

    (a) 1000

(b) 1200
(c) 1800
(d) 2400
(e) 2500

Answer: $\boxed{D}$

Solution: Let $DE = 3x$, $EF = 4x$, and $FD = 5x$. Then its area is $\frac{3x \cdot 4x}{2} = 6x^2$ and its perimeter is $12x$, so $12x = \frac{6x^2}{10}$, from which $x = 20$ and the area is $6 \cdot 20^2 = \boxed{2400}$ square units.

11. How many ordered triples (x, y, z) of positive integers have a product of 2022?

(a) 25
(b) 27
(c) 30
(d) 36
(e) 40

Answer: $\boxed{B}$

Solution: The factors of 2022 are 1, 2, 3, 6, 337, 674, 1011, and 2022. We then have the permutations of $(1, 1, 2022)$, $(1, 2, 1011)$, $(1, 3, 674)$, $(1, 6, 337)$, and $(2, 3, 337)$, of which there are $3 + 4 \cdot 3! = \boxed{27}$.

12. Triangle ABC has side lengths $AB = 6$, $BC = 8$, and $CA = 10$. A circle with center O is inscribed in $\triangle ABC$. What is the square of the length of OC?

(a) 20
(b) 28
(c) 32
(d) 38
(e) 40

Answer: $\boxed{E}$

Solution: The inscribed radius r of a triangle with semi-perimeter s satisfies the equation $A = rs$, where A is the area of the triangle. Since $A = \frac{6 \cdot 8}{2} = 24$ (because $\triangle ABC$ is a right triangle), and $s = \frac{6+8+10}{2} = 12$, we get $r = 2$. Then $OC^2 = (8-2)^2 + 2^2 = \boxed{40}$.

13. In how many ways can 16 be written as a sum of prime positive integers in non-increasing order? (For example, $7 + 5 + 2 + 2$ is one such representation.)

(a) 11
(b) 12
(c) 14
(d) 16
(e) 17

Answer: $\boxed{C}$

Solution: We have $13+3$, $11+5$, $11+3+2$, $7+7+2$, $7+5+2+2$, $7+3+3+3$, $7+3+2+2+2$, $5+5+3+3$, $5+2+2+2+2$, $5+3+3+3+2$, $5+3+2+2+2+2$, $3+3+3+3+2+2$, $3+3+2+2+2+2+2$, and $2+2+2+2+2+2+2+2$, for a total of $\boxed{14}$ ways. The more rigorous way to go about this is to utilize recursion: a 5 can be written as $3+2$, a 7 as $3+2+2$, etc.

14. If n is a positive integer for which $\sqrt{n+1} - \sqrt{n} < 0.1$, what is the smallest possible value of n?

   (a) 24
   (b) 25
   (c) 26
   (d) 49
   (e) 50

   Answer: $\boxed{B}$

   Solution: Write the inequality as

$$\sqrt{n+1} < \sqrt{n} + 0.1,$$

   and square both sides to obtain

$$n + 1 < n + 0.01 + 0.2\sqrt{n}.$$

   Thus, $0.99 < 0.2\sqrt{n}$, from which $\sqrt{n} > 4.95$ and $n \geq \boxed{25}$.

15. Albert and George are playing a very large number of games of rock-paper-scissors. So far, Albert has won 50.5 percent of the games, and George has won the remaining 49.5 percent. If George were to somehow win the next 400 games in a row (!), Albert will have won 49.5 percent of all the games played up to that point, while George will retake the lead with 50.5 percent of the games won. How many games have they played so far, not including the 400 games that would put George in the lead?

   (a) 19200
   (b) 19600
   (c) 19800
   (d) 20000
   (e) 20200

   Answer: $\boxed{C}$

   Solution: Say Albert has won $101n$ games while George has won $99n$ games. When George wins the next 400 games in a row, we will have $101n = \frac{99}{101}(99n + 400)$, or $101n = \frac{9801}{101}n + \frac{39600}{101}$. Hence, $10201n = 9801n + 39600$, so that $400n = 39600$ and $n = 99$. Therefore, Albert and George have played $200n = \boxed{19800}$ games so far.

16. A magical coin has the property that all flips until the point when at least one heads and at least one tails have been flipped have probability $\frac{1}{2}$ of coming up heads and probability $\frac{1}{2}$ of coming up tails. After both heads and tails have come up at least once, the probability of the coin coming up heads is $\frac{h}{h+t}$ and the probability of it coming up tails is $\frac{t}{h+t}$, where h and t are the numbers of heads and tails flipped up to that point, respectively. What is the probability that, out of 5 flips, exactly 4 are heads?

   (a) $\frac{5}{32}$
   (b) $\frac{3}{16}$
   (c) $\frac{7}{32}$
   (d) $\frac{17}{64}$
   (e) $\frac{9}{32}$

Answer: $\boxed{D}$

Solution: The probabilities of both THHHH and HTHHH are $\frac{1}{4} \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{16}$. The probability of HHTHH is $\frac{1}{8} \cdot \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{16}$. The probability of HHHTH is $\frac{1}{16} \cdot \frac{3}{4} = \frac{3}{64}$, and finally the probability of flipppping HHHHT is $\frac{1}{32}$. Altogether, this gives a probability of $\boxed{\dfrac{17}{64}}$.

17. Let r and s be the roots of $x^2 + 20x + 22$, and let t and u be the roots of $x^2 + 20x + 23$. What is the value of $rt + ru + st + su$?
    (a) 200
    (b) 305
    (c) 400
    (d) 485
    (e) 548

    Answer: $\boxed{C}$

    Solution: This is $(r + s)(t + u) = (-20)(-20) = \boxed{400}$ by Vieta's formulas (which give $r + s = t + u = -20$).

18. The number 2023 has the property that all of its nonzero digits have a product of 12. How many four-digit positive integers have this property?
    (a) 63
    (b) 66
    (c) 72
    (d) 78
    (e) 81

    Answer: $\boxed{E}$

    Solution: We have the permutations of 1126, 1134, 1026, 1034, 2023, 2006, and 3004, of which there are $12 + 12 + 18 + 18 + 9 + 6 + 6 = \boxed{81}$.

19. What is the sum of the base-16 logarithms of all of the positive integer divisors of 4096? Express your answer as a common fraction.
    (a) $\frac{55}{4}$
    (b) $\frac{33}{2}$
    (c) $\frac{39}{2}$
    (d) $\frac{91}{4}$
    (e) $\frac{105}{4}$

    Answer: $\boxed{C}$

    Solution: Recall that $\log_b(x) + \log_b(y) = \log_b(xy)$. Converting all of the divisors of $4096 = 16^3 = 2^{12}$, which are the powers of 2 from 2^0 to 2^{12} inclusive, to powers of 16, we get $16^{\frac{k}{4}}$ for $0 \leq k \leq 12$. Summing the exponents (since $a^m \cdot a^n = a^{m+n}$) gives $16^{\frac{78}{4}}$, the base-16 logarithm of which is $\boxed{\dfrac{39}{2}}$.

20. For every real number x, let $\lfloor x \rfloor$ be the greatest integer less than or equal to x. For example, $\lfloor 5.99 \rfloor = 5$ and $\lfloor -\pi \rfloor = -4$. What is the smallest positive integer n such that

$$\lfloor \sqrt{1} \rfloor + \lfloor \sqrt{2} \rfloor + \lfloor \sqrt{3} \rfloor + \cdots + \lfloor \sqrt{n} \rfloor \geq 100?$$

(a) 30
(b) 37
(c) 44
(d) 51
(e) 58

Answer: $\boxed{A}$

Solution: Up to 3, the floors are 1; up to 8, they are 2; up to 15, they are 3, and so forth. We have a sum of $3 \cdot 1 + 5 \cdot 2 + 7 \cdot 3 + 9 \cdot 4 = 70$ up to $n = 24$, so we need six more 5 terms, which gives $n \geq \boxed{30}$.

21. Rectangle $ABCD$ has positive integer side lengths r and s. Points E and F trisect $\overline{AB}$, points G and H trisect $\overline{BC}$, points I and J trisect $\overline{CD}$, and points K and L trisect $\overline{DA}$. If the area of octagon $EFGHIJKL$ is 2023, compute the number of possible ordered pairs (r, s).

(a) 8
(b) 9
(c) 10
(d) 12
(e) 15

Answer: $\boxed{B}$

Solution: The octagon is the rectangle with four triangles cut out, each of which has side lengths $\frac{1}{3}$ of those of the rectangle. Hence, the octagon's area is $1 - 4 \cdot \frac{1}{2} \cdot \left(\frac{1}{3}\right)^2 = \frac{7}{9}$ times that of the rectangle. As 2023 is $\frac{7}{9}$ of $289 \cdot 9 = 17^2 \cdot 3^2$, we want to count the divisors, of which there are $\boxed{9}$.

22. How many ordered tuples of positive integers have the property that the sum of the squares of the integers is 10?

(a) 15
(b) 16
(c) 18
(d) 20
(e) 21

Answer: $\boxed{B}$

Solution: We have the permutations of $(3, 1)$, $(2, 2, 1, 1)$, 2 and six 1s, and ten 1s, of which there are $2 + \binom{4}{2} + 7 + 1 = \boxed{16}$.

23. Rectangle $ABCD$ has side lengths $AB = 4$ and $BC = 5$. Point E is the midpoint of $\overline{AB}$, and point F lies on $\overline{BC}$ such that $BF = 4$ and $FC = 1$. Line segments $\overline{AF}$ and $\overline{DE}$ intersect at point G. What is $\frac{AG}{AF}$? Express your answer as a common fraction.

(a) $\frac{1}{3}$

(b) $\frac{5}{14}$
(c) $\frac{2}{5}$
(d) $\frac{7}{16}$
(e) $\frac{9}{20}$

Answer: $\boxed{B}$

Solution: Let $A = (0, 5)$, $B = (4, 5)$, $C = (4, 0)$, and $D = (0, 0)$, without loss of generality. Then $E = (2, 5)$ while $F = (4, 1)$, so $\overline{AF}$ has equation $x + y = 5$ while $\overline{DE}$ has equation $y = \frac{5}{2}x$. These intersect where $x = \frac{10}{7}$ and $y = \frac{25}{7}$, so that $\frac{AG}{AF} = \boxed{\frac{5}{14}}$.

24. For each positive integer n, let $Z(n)$ be the number of 0s in which n ends. For example, $Z(20220000) = 4$ and $Z(10001) = 0$. What is the sum of $Z(n)$ over all positive integer factors of 10^{10}?

(a) 120
(b) 165
(c) 220
(d) 286
(e) 325

Answer: $\boxed{C}$

Solution: Each factor is of the form $n = 2^i 5^j$, where $0 \le i, j \le 10$, and $Z(n) = \max(i, j)$, since we need both a 2 and 5 to form a factor of 10. Hence, our sum is $1 \cdot 10 + 2 \cdot 9 + 3 \cdot 8 + \cdots + 10 \cdot 1 = \boxed{220}$.

25. How many ordered triples of positive integers have a least common multiple of exactly 6?

(a) 42
(b) 45
(c) 47
(d) 49
(e) 50

Answer: $\boxed{D}$

Solution: We have $4^3 = 64$ possible triples, from the triples consisting of elements 1, 2, 3, and 6 which are the positive factors of 6. Of these, $2^3 - 1 = 7$ have only 1s and 3s, and not all 1s, hence an LCM of 3, $2^3 - 1 = 7$ have only 1s and 2s, and not all 1s, hence an LCM of 2, and one is the triple of all 1s. This gives $64 - 7 - 7 - 1 = \boxed{49}$ triples with an LCM of 6.

26. A geometric progression has a non-integer (but rational) common ratio, and has 7 terms, all of which are all integers. If not every term is a multiple of the last term, what is the smallest possible sum of all terms in the geometric progression?

(a) 254
(b) 665
(c) 1093
(d) 2059
(e) 2186

Answer: $\boxed{D}$

Solution: The common ratio cannot have a numerator of 1, since not every term is a multiple of the last term. Hence, we want it to be $\frac{2}{3}$, because we want the denominator to be as small as possible. This means that the last term may be as small as $3^6 = 729$, from which we get a sum of $729 + 486 + 324 + 216 + 144 + 96 + 64 = \boxed{2059}$.

27. How many permutations of $(1, 2, 3, 4, 5)$ have three consecutive elements a, b, and c, in that order, for which $a + b = c$?

    (a) 16
    (b) 20
    (c) 22
    (d) 28
    (e) 34

    Answer: $\boxed{C}$

    Solution: The consecutive elements may be 1, 2, and 3; 1, 3, and 4; 1, 4, and 5; or 2, 3, and 5. There are 3 positions to place the consecutive elements (positions 1-3 from left to right, positions 2-4, or positions 3-5), and 2 ways to order the other two elements. However, we have double-counted the permutations 12354 and 41235, so we actually have $4 \cdot 6 - 2 = \boxed{22}$ permutations.

28. Two complex numbers x and y satisfy the equations

$$x^2 + y^2 = 3,$$
$$x^2 + 2xy - y^2 = 7.$$

    Given that it is positive, what is the imaginary part of y^2? Express your answer as a common fraction in simplest radical form.

    (a) $\frac{\sqrt{17}}{4}$
    (b) $\frac{3\sqrt{3}}{4}$
    (c) $\frac{\sqrt{31}}{4}$
    (d) $\frac{\sqrt{33}}{4}$
    (e) $\frac{\sqrt{65}}{4}$

    Answer: $\boxed{C}$

    Solution: We have $x^2 + y^2 = (x + y)^2 - 2xy = 3$, and $x^2 + 2xy - y^2 = (x + y)^2 - 2y^2 = 7$. Hence, $2xy = 2y^2 + 4$, from which $xy = y^2 + 2$, $x = y + \frac{2}{y}$, and $x^2 = y^2 + 4 + \frac{4}{y^2}$. Thus, $2y^2 + 4 + \frac{4}{y^2} = 3$, and $2y^4 + y^2 + 4 = 0$, upon which substituting $z := y^2$ yields $2z^2 + z + 4 = 0$ and $z = \frac{-1 \pm i\sqrt{31}}{4}$, whose imaginary part is $\boxed{\dfrac{\sqrt{31}}{4}}$ (taking the positive value, as per the problem statement).

29. Let x and y be positive real numbers chosen uniformly at random from the set of ordered pairs (x, y) of positive real numbers such that $x + y = 3$. What is the expected value of the greatest integer not exceeding xy? Express your answer as a common fraction in simplest radical form.

    (a) $\frac{\sqrt{3}+1}{3}$
    (b) $\frac{\sqrt{5}+1}{3}$

(c) $\frac{2\sqrt{7}-3}{2}$

(d) $\frac{\sqrt{5}+3}{3}$

(e) $\frac{2\sqrt{13}-1}{2}$

Answer: $\boxed{B}$

Solution: We have $y = 3 - x$. If $x(3-x) < 1$, or $x^2 - 3x + 1 < 0$, then $0 < x < \frac{3-\sqrt{5}}{2}$. By symmetry, also, $\frac{3+\sqrt{5}}{2} < x < 3$, which are the intervals for which 0 is the largest integer not exceeding xy. This integer is 1 whenever $1 \le x(3-x) < 2$, or $\frac{3-\sqrt{5}}{2} \le x < 1$ or $2 < x \le \frac{3+\sqrt{5}}{2}$. Finally, this integer is 2 when $1 \le x \le 2$, since the maximum value of $x(3-x)$ is 2.25, attained at $x = 1.5$. It follows that the expected value of this integer is $\frac{1\cdot(\sqrt{5}-1)+2\cdot 1}{3} = \boxed{\dfrac{\sqrt{5}+1}{3}}$.

30. Let triangle ABC have side lengths $AB = 7$, $BC = 9$, and $CA = 8$. Point D lies on $\overline{BC}$ with $\tan(m\angle BAD) = 2\tan(m\angle DAC)$. The value of $\tan(m\angle DAC)$ can be written in the form $\frac{\sqrt{p}-\sqrt{q}}{r}$, where p, q, and r are positive integers and neither p nor q is a multiple of the square of a prime number. What is $p+q+r$?

(a) 18

(b) 21

(c) 32

(d) 62

(e) 70

Answer: $\boxed{E}$

Solution: We have

$$\tan(m\angle BAC) = \tan(m\angle BAD + m\angle DAC) = \frac{x+2x}{1 - x\cdot 2x} = \frac{3x}{1-2x^2}.$$

To compute $\tan(m\angle BAC)$, we use the area formula $\frac{1}{2}ab\sin(C)$ to get that $\sin(m\angle BAC) = \frac{3}{7}\sqrt{5}$, and so $\tan(m\angle BAC) = \frac{3}{2}\sqrt{5}$. Solving for x in the equation

$$\frac{3x}{1-2x^2} = \frac{3}{2}\sqrt{5},$$

we get

$$x^2(2\sqrt{5}) + 2x - \sqrt{5} = 0,$$

or $x = \frac{-1\pm\sqrt{11}}{2\sqrt{5}}$, the positive value which is $\frac{\sqrt{55}-\sqrt{5}}{10}$. As such, $p+q+r = 55+5+10 = \boxed{70}$.

Section 2: Short Answer

31. A positive integer with two digits is a multiple of 3, and its digits sum to a prime number. What is the sum of the possible values of this integer?

    Answer: $\boxed{63}$

    Solution: The digit sum is a multiple of 3, so it must be exactly 3. Then the integer can be 12, 21, or 30, and these sum to $\boxed{63}$.

32. How many integers from 0 through 7, inclusive, could be the remainder when the sum of two (not necessarily different) positive integer perfect squares is divided by 8?

    Answer: $\boxed{5}$

    Solution: A perfect square must leave a remainder of 0, 1, or 4 when divided by 8; this is because $(8k)^2 = 64k^2 \equiv 0 \bmod 8$, $(8k+1)^2 = 64k^2 + 16k + 1 \equiv 1 \bmod 8$, $(8k+2)^2 = 64k^2 + 32k + 4 \equiv 4 \bmod 8$, $(8k+3)^2 = 64k^2 + 48k + 9 \equiv 1 \bmod 8$, $(8k+4)^2 = 64k^2 + 64k + 16 \equiv 0 \bmod 8$, and likewise for $(8k+5)^2$, $(8k+6)^2$, and $(8k+7)^2$ by symmetry. Thus, 0, 1, 2, 4, and 5 could be remainders when the sum of two perfect squares is divided by 8, for $\boxed{5}$ possible values.

33. For how many positive integers n does 2022 leave a remainder of 2 when divided by n?

    Answer: $\boxed{10}$

    Solution: This is the number of factors of $2022 - 2 = 2020 = 2^2 \cdot 5 \cdot 101$, which is $(2+1)(1+1)(1+1)$, excluding 1 and 2, so we get $\boxed{10}$.

34. Let $d(n)$ be the units digit of n. What is the value of

    $$d(1^2) + d(2^2) + d(3^2) + \cdots + d(2022^2)?$$

    Answer: $\boxed{9095}$

    Solution: The units digits of the positive perfect squares up to 10^2 are $1, 4, 9, 6, 5, 6, 9, 4, 1, 0$, in that order, repeating every ten. Thus, our sum is $(1 = 4+9+6+5+6+9+4+1+0) \cdot 202 + 1 + 4 = \boxed{9095}$.

35. A fair six-sided die is labeled with the positive integers from 1 through 6, inclusive. What is the expected value of the square of a single roll? Express your answer as a common fraction.

    Answer: $\boxed{\frac{91}{6}}$

    Solution: This is $\frac{1^2+2^2+3^2+4^2+5^2+6^2}{6} = \boxed{\frac{91}{6}}$, since all rolls are equally likely.

36. Rectangle $ABCD$ has side lengths $AB = 4$ and $BC = 3$. Point E lies in the plane of $ABCD$, on the same side of $\overline{AC}$ as point B, so that $\overline{AE} \perp \overline{AC}$ and $AE = 10$. What is the length of line segment $\overline{DE}$? Express your answer in simplest radical form.

    Answer: $\boxed{\sqrt{157}}$

    Solution: The horizontal and vertical distances from A to E are 6 and 8, respectively, and the rectangle's height is 3, so the distance is $\sqrt{6^2 + (8+3)^2} = \boxed{\sqrt{157}}$.

37. In how many ways can we choose two distinct positive integers from the set $\{1, 2, 3, \cdots, 100\}$ that sum to a multiple of 33?

Answer: $\boxed{150}$

Solution: For sums of 33, 66, and 99, we have $(1, 32)$ through $(16, 17)$, $(1, 65)$ through $(32, 34)$, and $(1, 98)$ through $(49, 50)$, respectively. For 132, 165, and 198, we have $(32, 100)$ through $(65, 67)$, $(65, 100)$ through $(82, 83)$, and $(98, 100)$, respectively. This gives a total of $16 + 32 + 49 + 34 + 18 + 1 = \boxed{150}$ ways.

38. If x and y are positive real numbers such that $x^2+y^2 = 23$ and $x^3+y^3 = 110$, what is the value of x^4+y^4?

Answer: $\boxed{527}$

Solution: We have $(x + y)^2 - 2xy = 23$ and $(x + y)(x^2 - xy + y^2) = (x + y)((x + y)^2 - 3xy) = (x+y)(23-xy) = 110$. If $x+y = a$ and $xy = b$, then $a^2 - 2b = 23$ and $a(23-b) = 110$, so $b = 23 - \frac{110}{a}$ and $a^2 - 46 + \frac{220}{a} = 23$, or $a^3 - 69a + 220 = 0$. Then, by the rational root theorem, $a = 5$ is a root, and we factor out an $a - 5$ term (by setting the quotient equal to some arbitrary quadratic $a^2 + ba + c$) to get $a^2 + 5a - 44 = 0$, or $a = \frac{-5\pm\sqrt{201}}{2}$. As $x + y > 0$ and $xy > 0$, since x and y are both positive, we can only have $x + y = 5$, since even $\frac{-5+\sqrt{201}}{2} < \sqrt{23}$. This can be verified by squaring both sides, so that we get $\frac{226-10\sqrt{201}}{4} < 23$, or $10\sqrt{201} > 134$. This is true, because $20100 > 134^2 = 17956$. Thus, $xy = 1$, and $x^4 + y^4 = (x^2 + y^2) - 2(xy)^2 = 23^2 - 2 = \boxed{527}$.

39. Two distinct integers from 1 through 10, inclusive, are chosen uniformly at random. What is the expected value of their greatest common factor? Express your answer as a common fraction.

Answer: $\boxed{\frac{67}{45}}$

Solution: There are $\binom{10}{2} = 45$ total pairs of integers that can be chosen. Of these, $\binom{5}{2} - 1 = 9$ have a GCF of 2, 1 has a GCF of 4 ($(4, 8)$), $\binom{3}{2} = 3$ have a GCF of 3, and one ($(5, 10)$) has a GCF of 5. The rest have GCF 1, so the expected GCF is $\frac{31\cdot1+9\cdot2+3\cdot3+1\cdot4+1\cdot5}{45} = \boxed{\frac{67}{45}}$.

40. For some positive integer n, the triangle with side lengths 2, $\sqrt{17}$, and $\sqrt{n}$ has area 1. What is the sum of all possible values of n?

Answer: $\boxed{42}$

Solution: Without loss of generality, say the triangle has vertices at $(0, 0)$ and $(2, 0)$. Observe that $17 = 4^2 + 1^2$, and that the triangle must have height 1 in order to have area 1, so this is the only viable way to split 17 into a sum of squares. We then have $n^2 = (2 - 4)^2 + 1^2 = 5$ or $n^2 = (2 + 4)^2 + 1^2 = 37$, so the sum of the possible values of n is $\boxed{42}$.

41. (Source: BmMT 2021) What is the sum of $\frac{1}{n}$ over all positive integer factors n of 360? Express your answer as a common fraction.

Answer: $\boxed{\frac{13}{4}}$

Solution: Say a factor n of 360 is equal to $\frac{360}{k}$, where k is an integer that is itself a factor of 360. Then $\frac{1}{n} = \frac{k}{360}$, and so we want to sum $\frac{k}{360}$ over all factors k of 360. This is $\frac{1}{360}$ the sum of factors of $360 = 5 \cdot 3^2 \cdot 2^3$, which is $(5 + 1)(3^2 + 3 + 1)(2^3 + 2^2 + 2 + 1) = 1170$. Simplifying the quantity $\frac{1170}{360}$, we get $\boxed{\frac{13}{4}}$ as the desired sum.

42. How many sequences of positive integers in non-decreasing order have a last term no larger than 10?

Answer: $\boxed{184766}$

Solution: Suppose the sequence has length $2 \le l \le 10$ with last term t. Then we want to add a total n (such that the first term is $t-n$) that is at most 9 in the form of $l-1$ non-negative integer additions, which by stars-and-bars can be done in $\binom{n+(l-1)-1}{l-2} = \binom{n+l-2}{l-2}$ ways. Summing over $2 \le l \le 10$, $1 \le t \le 10$, and $0 \le n \le t-1$, we want to compute

$$10 + \sum_{l=2}^{10} \sum_{t=1}^{10} \sum_{n=0}^{t-1} \binom{n+l-2}{l-2}$$

$$= 10 + \sum_{l=2}^{10} \sum_{t=1}^{10} \binom{t+l-2}{l-1}$$

$$= 10 + \sum_{l=2}^{10} \sum_{t=1}^{10} \binom{t+l-2}{l-1}$$

by the Hockey-Stick identity, which in turn simplifies to

$$= 10 + \sum_{l=2}^{10} \binom{l+9}{l}$$

$$= 10 + \sum_{l=2}^{10} \binom{l+9}{9}$$

$$= 10 + \binom{20}{10} = \boxed{184766}.$$

43. Triangle ABC has $AB = 13$, $BC = 14$, and $CA = 15$. Points D and E lie on $\overline{AB}$ and $\overline{CD}$, respectively, such that $m\angle DAE = m\angle EAC$. If $DE = 2$, what is the largest integer less than or equal to 10 times the length of $\overline{AD}$?

Answer: $\boxed{25}$

Solution: Let $AD = x$; then by the angle bisector theorem, $DC = 2 + \frac{30}{x}$. It's also equal to $\sqrt{\left(12 \cdot \frac{13-x}{13}\right)^2 + \left(9 + \frac{5}{13}x\right)^2}$, since we can drop the altitude with length 12 from A to $\overline{BC}$ to get a 5-12-13 right triangle and a 9-12-15 right triangle. If $x = 2.6$, then the LHS is $\frac{176}{13} \approx 13.54$, while the RHS is $\sqrt{\frac{2304}{25} + 100} \approx 13.86$. (These can be reasonably approximated to 1 decimal place.) However, for $x = 2.5$, the LHS is 14, while the RHS is certainly smaller than 13.86, so $2.5 < AD < 2.6$, and the answer is $\boxed{25}$.

44. Let B be the answer to question 45. In how many ways (with respect to order) can we construct a list of three elements chosen from either B or C, such that their sum is an odd multiple of 3? ($(2, 2, 5)$ and $(5, 2, 2)$ are two different lists, for example.)

Answer: $\boxed{4}$

45. Let C be the answer to question 46. What is the geometric mean of A and C?

Answer: $\boxed{6}$

46. Let A be the answer to question 44. What is the largest positive integer k for which 2^k is a factor of A^3B^3?

Answer: $\boxed{9}$

Solution to questions 44-46: Note that, in question 46, the answer cannot be zero, so at least one of A and B is even. Begin by noting that, if exactly one of B or C is odd, then $A = 4$, but only if the even number is congruent to the other modulo 3, since we can have odd + odd + odd or odd + even + even (in three different ways). If both are even, then $A = 0$, and if both are odd and congruent mod 3, then $A = 8$, and otherwise, $A = 2$.

If B is even, then AC is a multiple of 4. If C is odd, then, A must be a multiple of 4, and this also means that $C \geq 6$. Since $\sqrt{AC}$ is a positive integer, also noting that C is a multiple of 3, with $A = 4$, we get $C = 9$ and $B = 6$, which works. If $A = 8$, then $\sqrt{AC}$ cannot be even without C also being even.

If we instead assume C is even, then AB is a multiple of 4, and $ABC = B^2$ is a multiple of 8. This means that B must be a multiple of 4, and AC is actually a multiple of 16. As C is a multiple of 3, $AC = B^2$ is a multiple of 48, so B is a multiple of 12. This means AC is a multiple of 144. But since $A \leq 8$, C must be at least 18, implying that AB is divisible by 64. Hence, B must be divisible by 8, and thus, by 24, meaning AC is actually divisible by 576, and C is divisible by 72. In turn, AB is divisible by 2^{24}, B is divisible by 2^{21}, AC is a multiple of 2^{42}, and so on, in an infinite descent. Thus this case cannot happen.

On the other hand, if we instead suppose that B is odd, then $AC = B^2$ must be odd, and A and C are both odd as a result. Thus, AB is odd and $C = 0$, but this is a contradiction! Hence $(A, B, C) = \boxed{(4, 6, 9)}$ is the unique solution.

Section 3: Free Response

47. **[35]** Definitions and their applications.

(a) **[3]** Please state the Binomial Theorem, and use it to find the coefficient of x^5y^2 in

$$(3x + 2y)^7.$$

Solution: The binomial theorem, in its simplest and most familiar form, says that

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Unpacking this notation for a bit: we define the *choose function*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

which is the number of ways to choose k objects out of a total of n distinguishable objects, without regard to order. The theorem holds, because in expanding out the left-hand side, we can "choose" k powers of x and $n - k$ powers of y to multiply to an $x^k y^{n-k}$ term in $\binom{n}{k}$ ways.

More generally, we also have

$$(ax + by)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} x^k b^{n-k}$$

which can be seen through the same reasoning. We also have Newton's binomial theorem, which extends this to negative and/or fractional powers. The coefficient of x^5y^2 in $(3x + 2y)^7$ is then $\binom{7}{2}3^5 2^2 = \boxed{20412}$.

(b) **[3]** What is the Pigeonhole Principle? Please use it to show that, for any three positive integers, the sum of some two of them must always be even.

Solution: The Pigeonhole Principle states that, whenever we are sorting n objects into k disjoint groups, and $n > k$, at least one of the groups must contain two or more objects. In particular, one group must contain at least $\left\lfloor \frac{n}{k} \right\rfloor$ objects. This is because, if we tried to fill each group with as few objects as possible in each group, we'd run out of space after k objects, but we'd still have some left over – hence, one woud need to go in a group that already has an object, making it have 2 or more objects after all of them are sorted.

Suppose we had three positive integers a, b, and c, each of which is either even or odd. The pigeonhole principle (here with $n = 3$ and $k = 2$; the objects are the integers, and the groups/holes are the different possible parities of a positive integer, of which there are two – odd and even) says that some two of the integers must either be both even or both odd. Either way, we can sum the two integers that are of the same parity together to get an even number (since even + even = even, and odd + odd = odd).

(c) **[4]** Please explain the combinatorial concept of *stars and bars,* and apply it to solve a problem of your choosing.

Solution: The *stars-and-bars* method applies to a situation where we want to sort some number n of indistinguishable objects into some number k of indistinguishable bins. The method gets its name from the situation where we have a certain number of "stars," or indistinguishable objects, and want to separate them by way of a certain number of "bins" into a number of disjoint groups.

For example, say we wanted to find the number of ordered triples of positive integers summing to 7. An equivalent problem is to find the number of ordered triples of *non-negative* integers summing to 4. By stars-and-bars, this is the same as placing two bars (since we want three groups, we need two separators, as each separator forms a new group) between 4 stars, where each "star" represents an "action" of adding 1 to one of the three integers, beginning with 0/0/0. How many ways can we do this? That is, how many ways can we arrange 4 stars and 2 bars in one row of 6 objects? It's just $\binom{6}{2} = 15$. This, of course, assumes we can have two bars right next to each other, which we can: this variant of the problem says nothing about not being able to have an empty group or groups (i.e. one or two of the integers being equal to 0).

In general, what if we had k bars to place between n stars? There would be a total of $n+k$ objects, and k places to put the bars, hence $\binom{n+k}{k}$ ways to arrange all of the objects. We'd actually have $k+1$ groups if we did this, though, so we tend to use the formula $\binom{n+k-1}{k-1}$ in reference to having k *groups* (such as numbers of positive integers) and n stars. And in our previous example – with $n = 4$ and $k = 3$ (that is, 3 non-negative integers that sum to 4), does this formula work? Yes – we get $\binom{4+3-1}{3-1} = 15$ as earlier, so it looks like everything checks out!

(d) **[5]** Please state Vieta's formulas for a quadratic polynomial. How can we calculate the sum of the squares of the roots of a quadratic polynomial? What about the sum of the fourth powers? Can you generalize this to polynomials of arbitrary degree?

Solution: Vieta's formulas state that the sum and the product of the roots of $ax^2 + bx + c$ ($a \neq 0$) are $-\frac{b}{a}$ and $\frac{c}{a}$, respectively. These can be proven by a direct application of the quadratic formula, and/or by noting that the axis of symmetry of this parabola is $x = -\frac{b}{2a}$, and that the roots are symmetric about this axis. Denote by r and s the roots of this polynomial. To compute $r^2 + s^2 = (r+s)^2 - 2rs$, we calculate $\frac{b^2}{a^2} - \frac{2c}{a} = \frac{b^2-2ac}{a^2}$. Similarly, we compute $r^4 + s^4 = (r^2+s^2)^2 - 2r^2s^2$. For a generalization, we can turn to Newton's sums, which allow us to define recurrence relations in the sums S_k of the k^{th} powers of the roots of a polynomial. Where we have a polynomial

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

with roots $x_1, x_2, x_3, \cdots, x_n$, we have the recurrence relations

$$\begin{aligned} a_n S_1 + a_{n-1} &= 0, \\ a_n S_2 + a_{n-1}S_1 + a_{n-2} &= 0, \\ a_n S_3 + a_{n-1}S_2 + a_{n-2}S_1 + a_{n-3} &= 0, \\ &\vdots \end{aligned}$$

where $a_i = 0$ whenever $i < 0$. To prove that these hold (although certainly not necessary for full credit), we substitute each root into the polynomial and get 0 each time, then multiply each of the resulting n equations by x_1^{k-n} to get n equations in x_1^k, x_2^k, x_3^k, and so forth, which we then add together to obtain a single polynomial equation in $(x_1^k + x_2^k + x_3^k + \cdots + x_n^k)$. From this we obtain the general recurrence relation.

(e) **[6]** Please carefully state de Moivre's theorem. Why can we represent complex numbers in terms of sine and cosine? How do you think we might define sine and cosine for complex arguments?

Solution: de Moivre's theorem states that, for all real numbers x and positive integers n,

$$(\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx).$$

Because complex numbers $x + iy$ are (*a priori*) represented by points $(x, y) \in \mathbb{R}^2$, we can do the same with the quantity $\sin(x) + i\cos(x)$: the result is a vector from the origin to the point $(\sin(x), \cos(x))$, which will always have magnitude 1 (as $\sin^2(x) + \cos^2(x) = 1$ for all x). Indeed, this identity itself comes as a consequence of the unit circle, which consists of all points of the form $(\sin(x), \cos(x))$ for $0 \leq x < 2\pi$.

Because of this, and Euler's formula $\cos(x) + i\sin(x) = e^{ix}$, we can actually extend sine and cosine to complex arguments:

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i},$$
$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

(using the facts that $\sin(-z) = -\sin(z)$ and $\cos(-z) = \cos(z)$ for $z \in \mathbb{C}$).

(f) **[7]** Please define the *power* of a point with respect to a circle, and carefully state the *power of a point* theorem in all of its forms. Please sketch a proof of at least one of those forms.

Solution: With respect to a circle with center O and radius r, the *power* of the point P in the plane of the circle is $OP^2 - r^2$. The power of a point theorem, in its first form, says that for two chords $\overline{AC}$ and $\overline{BD}$ of a circle intersecting at point P inside the circle, we have $AP \cdot CP = BP \cdot DP$; in its second form, it says that for a tangent $\overline{AB}$ and secant $\overline{BD}$ of a circle meeting at the point B outside the circle, we have $AB^2 = BC \cdot BD$; and in its third form, it says that, for two secants $\overline{AC}$ and $\overline{CE}$ with B and D being the points of itnersection of these secants with the circle's circumference, we have $AC \cdot BC = CD \cdot CE$. For a proof of all three of these forms at once, it suffices to note that $\triangle ABP$ is similar to $\triangle CDP$, in particular drawing the line segment $\overline{AC}$.

(g) **[7]** What is Euler's phi/totient function, and why is it a multiplicative function? Please carefully state and prove Fermat's little theorem and Euler's totient theorem, and explain why FLT is a direct consequence of Euler's theorem.

Solution: Euler's totient function $\varphi : \mathbb{N} \to \mathbb{N}$ is defined such that $\varphi(n)$ is the number of positive integers less than or equal to n that are relatively prime to n. Recall the definition of a multiplicative function f (over the positive integers, without loss of generality): for relatively prime positive integer arguments m and n, $f(m)f(n) = f(mn)$. To prove that φ satisfies this property, consider a rectangular array of m rows and n columns, and label the cells with the positive integers from 1 through mn, inclusive. Now cross out any cell whose label has a factor of either m or n: we should end up with a sub-array that has dimensions $\varphi(m)$ by $\varphi(n)$. (Consider also the connection this has with the statement of the Chinese remainder theorem!)

Fermat's little theorem says that, for a positive integer a and prime number p, $a^p \equiv a \bmod p$, or equivalently, $a^{p-1} \equiv 1 \bmod p$. Euler's theorem is a generalization of FLT, in that for all n with $\gcd(a, n) = 1$, $a^{\varphi(n)} \equiv 1 \bmod n$ (clearly, this holds for prime p, since $\varphi(p) = p-1$, the only integer less than or equal to p *not* relatively prime with p being p itself).

We prove FLT by induction via the binomial theorem (though perhaps more elegant combinatorial methods certainly exist). The base case $a = 0$ is trivial; assuming $a^p \equiv a \bmod p$ for a not a multiple of p, we want to show that

$$(a+1)^p \equiv (a+1) \bmod p.$$

Expanding out the LHS, all terms contain a factor of p except for the very first and last terms, which are a^p and 1, respectively – hence, proved.

As for Euler's theorem, we can consider the set of all positive integers relatively prime to n, and multiply each element by some a relatively prime to n. We claim that the new set is a permutation of the original set, so that, when we multiply all the elements together, we end up with $a^{\varphi(n)} \equiv 1 \bmod n$, where $\varphi(n)$ is the cardinality of both sets by definition. This is because the set of positive integers less than or equal to n that are relatively prime to n is a group, hence closed under multiplication – in fact, the group $(\mathbb{Z}/n)^*$. (One can also prove this by Lagrange's theorem, which states that the order (number of elements) of a subgroup H of a group G divides the order of G.)

48. **[25]** Conceptual questions.

(a) **[6]** Explain why the following formulas for the area of a triangle are valid: $\frac{1}{2}bh$, $\frac{1}{2}ab\sin C$, Heron's formula, Shoelace formula.

Solution:

- $\frac{1}{2}bh$: Drop an altitude of the triangle, and note that each of the two resulting right triangles can be reflected over their diagonals to produce a rectangle with height h and base length b.
- $\frac{1}{2}ab\sin C$: For acute triangles with $C < 90°$, draw the altitude opposite the angle C, so that $\sin(C)$ is the ratio of that altitude length h to the length of $\overline{AC} =: b$. Then $\frac{1}{2}ab\sin C = \frac{1}{2}ah$, which by the previous part, is the area of the triangle.

  For obtuse triangles with $C > 90°$, we use the fact that $\sin(180° - x) = \sin(x)$ for all x to write $\sin(C) = \sin(180° - C)$, and repeat the above argument by drawing the altitude *outside* the triangle down to $\overline{BC}$.
- Heron's formula: Say a triangle has side lengths a, b, and c, and that the side of length b is "flat" (parallel to the x-axis). Let the height of the triangle (down to the side of length b) be h, and let the portion of the flat side length adjacent to side length a be x and the remaining portion (adjacent to side length c) be $b - x$. Then, by the Pythagorean theorem, we have the simultaneous equations $x^2 + h^2 = a^2$, $(b - x)^2 + h^2 = c^2$. Expanding gives $b^2 - 2bx + x^2 + h^2 = c^2$, so that $b^2 - 2bx = c^2 - a^2$ and $x = \frac{b^2+a^2-c^2}{2b}$. We want to express h in terms of a, b, and c, and we know that

  $$h^2 = a^2 - x^2 = a^2 - \frac{(b^2 + a^2 - c^2)^2}{4b^2} = \frac{4a^2b^2 - (b^4 + a^4 + c^4 + 2a^2b^2 - 2b^2c^2 - 2a^2c^2)}{4b^2}$$

  $$= \frac{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4}{4b^2}.$$

  Therefore, we have

  $$\frac{1}{2}bh = \frac{\sqrt{2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4}}{4},$$

  and this is equivalent to Heron's formula.
- Shoelace: Without loss of generality, say that the triangle has vertices at $(0, 0)$, (a, b), and (c, d), because any triangle can be translated in the xy-plane so that one of its vertices lies at the origin. Furthermore, assume wlog that $c \geq a$ and $b \geq d$. Then the triangle is inscribed in a rectangle with side lengths c and b, with right triangles cut out whose side lengths are a and b, c and d, and $c - a$ and $b - d$. We then get the area of the triangle as

  $$cb - \frac{ab + cd + (c - a)(b - d)}{2} = \frac{2cb - (ab + cd + (cb - cd - ab + ad)}{2} = \frac{cb - ad}{2}.$$

  As we wanted to show that the area was $\frac{1}{2}|ad - bc|$, we are done, since in this case, $bc \geq ad$, and $ad - bc < 0$, meaning that $|ad - bc| = cb - ad$ as desired.

(b) **[4]** Explain why the centroid cuts each median of a triangle into a 2:1 ratio.

Solution: Suppose we have triangle ABC with midpoints D of $\overline{BC}$, E of $\overline{AC}$, and F of $\overline{AB}$. Draw $\overline{EF}$; then $\triangle AEF$ is similar to $\triangle ACB$, and so $\overline{EF} \parallel \overline{CB}$ with $CB = 2EF$. Letting G be the centroid of $\triangle ABC$, we note that $\triangle EFG$ is similar to $\triangle CBG$, because of vertical angles ($m\angle GEF = m\angle GBC$, and $m\angle EFG = m\angle GCB$). Hence, $BG = 2GE$ and $CG = 2GF$. To prove $AG = 2GD$ likewise, draw the line segment $\overline{DF}$ and observe that triangles $\triangle DFG$ and $\triangle AGC$ are similar in the same way.

(c) **[5]** Can you prove the formulas for sine, cosine, and tangent addition/subtraction?

Solution: It suffices to prove these for addition; we can simply substitute $-y$ in place of y in the addition formulas $\sin(x+y)$, $\cos(x+y)$, and $\tan(x+y)$ (using the fact that $\cos(-y) = \cos(y)$ and $\sin(-y) = -\sin(y)$).

To prove that

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y),$$

we can consider a line segment $\overline{AB}$ (wlog with length 1) sweeping out an angle measure of x to line segment $\overline{AC}$, then another angle measure of y in the same direction to $\overline{AD}$. Drop the perpendiculars from D to E on $\overline{AB}$, from D to F on $\overline{AC}$, from F to G on $\overline{AB}$, and from F to H on $\overline{DE}$. Then observe that $\sin(x) = \frac{FG}{AF}$, $\cos(x) = \frac{DH}{DF}$ (since $m\angle HDF = x$), $\sin(y) = \frac{FD}{DA}$, and $\cos(y) = \frac{AF}{DA}$. Also note that $\sin(x+y) = \frac{DE}{AD}$ and $\cos(x+y) = \frac{AE}{AD}$, and that we obtain

$$\sin(x)\cos(y) + \cos(x)\sin(y) = \frac{FG}{AF} \cdot \frac{AF}{DA} + \frac{FD}{DA} \cdot \frac{HD}{DF} = \frac{FG + HD}{DA} = \frac{DE}{DA} = \sin(x+y).$$

The cosine addition formula is basically identical, except using the observation that $EA = GA - FH$.

Using the sin/cos addition formulas, we have

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)}$$

$$= \frac{\sin(x)\cos(y) + \cos(x)\sin(y)}{\cos(x)\cos(y) - \sin(x)\sin(y)}$$

$$= \frac{\left(\frac{\sin(x)\cos(y)+\cos(x)\sin(y)}{\cos(x)\cos(y)}\right)}{\left(\frac{\cos(x)\cos(y)-\sin(x)\sin(y)}{\cos(x)\cos(y)}\right)}$$

$$= \frac{\frac{\sin(x)}{\cos(x)} + \frac{\sin(y)}{\cos(y)}}{1 - \frac{\sin(x)}{\cos(x)} \cdot \frac{\sin(y)}{\cos(y)}}$$

$$= \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)},$$

as desired.

(d) **[10]** Building upon the previous part, the *Chebyshev polynomials of the first kind* are defined by

$$T_n(\cos(\theta)) := \cos(n\theta), \tag{1}$$

and the *Chebyshev polynomials of the second kind* are defined by

$$U_n(\cos(\theta)) := \frac{\sin((n+1)\theta)}{\sin(\theta)} \tag{2}$$

for all non-negative integers n and real numbers θ.

i. **[7]** Explain why we have linear recurrence relations for T_n and for U_n, and write explicitly what the recurrence relations are. Why do you think they're so similar? What does all of this have to do with de Moivre's formula?

ii. **[3]** The Chebyshev polynomials of the first kind are useful in approximation theory as the polynomials (with the largest possible leading coefficient) whose absolute value on $[-1,1]$ is at most 1. Because of this, their roots, which are called *Chebyshev nodes,* are used in polynomial interpolation of continuous functions; the resulting interpolant is a close approximation of the function. Intuitively speaking, where do you think this connection comes from?

Solution:

i. Because $T_n(\cos(\theta)) = \cos(n\theta)$, we have

$$T_{n+1}(\cos(\theta)) = \cos((n+1)\theta) = \cos(n\theta+\theta) = \cos(n\theta)\cos(\theta) - \sin(n\theta)\sin(\theta)$$

as well as

$$T_{n-1}(\cos(\theta)) = \cos((n-1)\theta) = \cos(n\theta-\theta) = \cos(n\theta)\cos(\theta) + \sin(n\theta)\sin(\theta),$$

so that

$$(T_{n+1}+T_{n-1})(\cos(\theta)) = 2\cos(n\theta)\cos(\theta),$$

and therefore,

$$2xT_n(x) = T_{n+1}(x) + T_{n-1}(x),$$

or, in other words,

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

The recurrence relation for U_n is similar, and its derivation is left as an exercise:

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x),$$

only with $U_1(x) = 2x$ instead of $T_1(x) = x$; the similarity comes from the fundamental similarity of the sine and cosine addition formulas. Recall that de Moivre's formula says that $(\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx)$, so if we can express the RHS in terms of the Chebyshev polynomials T_n and U_n, we've effectively found a way to expand trigonometric polynomials and relate them to powers of "basic" polynomials in $\sin(x)$ and $\cos(x)$ (see also the recursion formulas we've just proven).

ii. If we substitute $x := \cos(t)$, as we did in part (i) to obtain the recurrence relations, we can write a polynomial as a linear combination of terms including $\cos(nt)$ terms, which actually form the basis for the study of Fourier series. The ony constraint is on the coefficient of $\cos(nt)$, which give rise to the Chebyshev polynomials.

49. **[20]** Applied contest-like questions.

(a) **[4]** Compute the remainder when

$$\underbrace{202220222022\cdots 2022}_{\text{2022 copies of 2022}}$$

is divided by 33.

Answer: $\boxed{18}$

Solution: By the Chinese remainder theorem, we split this into mod 3 and mod 11. Modulo 3, we compute the sum of the digits as $6 \cdot 2022$, which is a multiple of 3, so we get 0 mod 3. Modulo 11, the alternating sum of digits is $4 \cdot 2022) - 2 \cdot 2022 = 4044 \equiv 7$ mod 11. Hence, we get $\boxed{18}$ mod 33.

(b) **[6]** (Source: SMT 2019) Let n be a real number. What is the maximum possible value of

$$|\sqrt{n^2+4n+5}-\sqrt{n^2+2n+5}|?$$

Answer: $\boxed{\sqrt{2}}$

Solution: Notice that $n^2+4n+5=(n+2)^2+1^2$, and $n^2+2n+5=(n+1)^2+2^2$. Consider the points $(0,0)$, $(n+2,1)$, and $(n+1,2)$. By the triangle inequality, the distance between the points $(n+2,1)$ and $(n+1,2)$, which is $\boxed{\sqrt{2}}$, is the maximum (equality holds).

(c) **[10]** Let r, s, and t be the complex roots of

$$x^3-x^2+x+1.$$

Compute

$$\frac{1}{r^3}+\frac{1}{s^3}+\frac{1}{t^3}.$$

Answer: $\boxed{-7}$

Solution: Consider the quantity

$$(r^2-r+1)(s^2-s+1)(t^2-t+1),$$

which is equal to

$$(r^3+2)(s^3+2)(t^3+2)=(rst)^3+2((rs)^3+(rt)^3+(st)^3)+4(r^3+s^3+t^3)+8,$$

and also to $\left(-\frac{1}{r}\right)\left(-\frac{1}{s}\right)\left(-\frac{1}{t}\right)=-\frac{1}{rst}=1$ by Vieta's formulas.

This means that

$$-1+2\left(-\frac{1}{r^3}-\frac{1}{s^3}-\frac{1}{t^3}\right)+4(r^3+s^3+t^3)+8=1,$$

or

$$-2\left(\frac{1}{r^3}+\frac{1}{s^3}+\frac{1}{t^3}\right)+4(r^3+s^3+t^3)=-6.$$

It remains to compute $r^3+s^3+t^3$. Since

$$r^3+s^3+t^3=(r+s+t)^3-3r^2(s+t)-3s^2(r+t)-3t^2(r+s)-6rst$$

$$=1-3r^2(1-r)-3s^2(1-s)-3t^2(1-t)+6=7-3(r^2+s^2+t^2)+3(r^3+s^3+t^3),$$

and

$$r^2+s^2+t^2=(r+s+t)^2-2(rs+st+tr)=1-2(1)=-1,$$

we have

$$r^3+s^3+t^3=10+3(r^3+s^3+t^3),$$

so that $r^3+s^3+t^3=-5$, and $\frac{1}{r^3}+\frac{1}{s^3}+\frac{1}{t^3}=\boxed{-7}$.