cyber math

Online Open Tournament

Contest File (Europe/Africa)

Middle School Division

Saturday, March 26, 2022

1. Assuming that 1 meter is equal to 100 centimeters, 1 yard is equal to 3 feet, 1 foot is equal to 12 inches, and 40 inches is equal to 1 meter, how many centimeters are equal to one yard?

2. Blanche writes, for each $1 \leq k \leq 26$, k copies of the k^{th} letter of the alphabet in a row, so that her string begins ABBCCCDDDD $\cdots$; and ends with 26 Z's. What is the middle letter in her string?

   (a) M
   (b) P
   (c) R
   (d) T
   (e) none of the above

3. The circumference of a $150°$ sector of a circle with integer radius is $15\pi + k$ for some integer k. What is k?

   (a) 18
   (b) 24
   (c) 36
   (d) 60
   (e) none of the above

4. The sum of the square roots of three distinct positive integers a, b, and c summing to 35 is an integer. Compute the product abc.

5. How many positive integers between 1 and 100, inclusive, are the positive difference between two numbers of the form $N^2 - N + 1$ for some positive integer N?

6. Sixty times a positive integer leaves a remainder of 58 when divided by 119. Compute the smallest possible value of this positive integer.

7. Triangle ABC has $AB = AC = 10$ and $BC = 12$. Point D lies on $\overline{BC}$ with $BD = 10$. Compute AD^2.

8. Chester flips 4 fair coins, and Rhiannon flips 6 fair coins. What is the probability that Rhiannon flips more heads than Chester? Express your answer as a common fraction.

9. Square $MATH$ has side length 2. Point P lies on $\overline{MA}$ such that the area of quadrilateral $PATH$ is 3.9. Compute the area of overlap between quadrilateral $PATH$ and triangle MTH. Express your answer as a common fraction.

10. Compute the sum of all integers n such that

$$\frac{\sqrt{n}+5}{\sqrt{n}+1}$$

    is an integer.

11. An isosceles triangle has a base length of 10 and two side lengths of 13. A point inside the triangle is equidistant with common distance d from all three vertices of the triangle. Then d can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

12. A positive integer is called *stable* if none of its digits are greater than the cube of the smallest digit. Compute the number of stable positive integers less than or equal to 1000.

13. Given that

$$7 < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{673} < 8,$$

    compute the largest integer not exceeding

$$\frac{3}{1+2} + \frac{6}{4+5} + \frac{9}{7+8} + \frac{12}{10+11} + \cdots + \frac{2022}{2020+2021}.$$

14. Let rectangle $ABCD$ have $AB = 3$ and $BC = 6$. Points E, F, G, and H lie on $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ respectively such that $EB = GD = 1$ and $EF = GH$. If the perimeter of $AEFCGH$ is at most 90 percent of the perimeter of $ABCD$, compute the maximum possible length of $\overline{FC}$. Express your answer as a common fraction.

15. The *digital root* of a positive integer is the result of repeatedly summing the digits of that integer until a single integer from 1 to 9, inclusive, is obtained. For example, the digital root of 2022 is 6, the digital root of 1234567 is 1 (since the sum of digits is 28, $2 + 8 = 10$, and $1 + 0 = 1$), and the digital root of 36 is 9. Compute the sum of the digital roots of all the positive integers from 1 to 2022, inclusive.

16. A set consists of sixteen distinct positive integers which sum to 139. When one of these sets is chosen uniformly at random, compute the expected value of its largest element.

17. For each positive integer n, define
$$f(n) = \frac{\sum_{i=1}^{n}(i + i^2)}{\sum_{i=1}^{n} i^3}.$$
Compute the smallest positive integer n for which $f(n) \leq \frac{1}{10}$.

18. Let $ABCD$ be a rectangle with $AB = 2$ and $BC = 1$. Suppose that E and F are points on $\overline{AB}$ and $\overline{CD}$, respectively, lying on the same side of $\overline{BC}$, such that $AE \cdot CF = 1$ and $EF = \frac{5}{4}$. The largest possible length of CF can be written in the form $\frac{p+\sqrt{q}}{r}$, where p, q, and r are positive integers with q not divisible by the square of a prime. Compute $p + q + r$.

19. Let n be a positive integer. A permutation of $(a_1, a_2, a_3, \cdots, a_{2n})$ is called *rightweight* if $2(a_1 + a_2 + a_3 + \cdots + a_n) \leq a_{n+1} + a_{n+2} + a_{n+3} + \cdots + a_{2n}$. Compute the number of permutations of $(1, 2, 3, 4, 5, 6, 7, 8)$ that are rightweight.

20. Let $\tau(n)$ denote the number of positive integer divisors of n. Compute the number of positive integers $n \leq 100$ satisfying $\tau(n) + \tau(2n) = 18$.