cyber math

Online Open Tournament

Contest Solutions (Asia/Australia)

Middle School Division

Saturday, March 26, 2022

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1. How many positive integer factors of 24 are even?

   (a) 4
   (b) 6
   (c) 7
   (d) 8
   (e) none of the above

   Answer: $\boxed{B}$

   Writing $24 = 2^3 \cdot 3$, we notice that an even factor has a nonzero exponent of 2 in its prime factorization. We have 3 choices for this power of 2 and 2 choices for the power of 3, giving us $3 \cdot 2 = \boxed{6}$ even factors of 24. (It is also not too hard to list out all the factors directly and count the ones that are even.)

2. At a stationery store, 3 pencils cost 70 cents, but 25 pencils cost only 470 cents. Compute the amount, in cents, one would save by buying 75 pencils in 3 packs of 25 instead of 25 packs of 3.

   Answer: $\boxed{340}$

   3 packs of 25 pencils each cost $470 \cdot 3 = 1410$ cents, while 25 packs of 3 pencils each would cost $70 \cdot 25 = 1750$ cents; one would save $1750 - 1410 = \boxed{340}$ cents.

3. Eight fair coins are flipped at the same time. The most likely outcome has a probability $\frac{p}{q}$ of occurring, where p and q are relatively prime positive integers. Compute $p + q$.

   Answer: $\boxed{163}$

   The probability of h heads coming up is $\frac{\binom{8}{h}}{2^8}$, which is maximized for $h = 4$ (the choose function is symmetric about the middle). This evaluates to $\frac{\binom{8}{4}}{2^8} = \frac{35}{128}$, and $p + q = 35 + 128 = \boxed{163}$.

4. Suppose that $x = -\frac{9}{2}$ is the unique real solution to the equation $x^2 + ax + b = 0$. Compute $a + b$.

   Answer: $\boxed{\dfrac{117}{4}}$

   Note that $x^2 + ax + b = \left(x + \frac{9}{2}\right)^2 = x^2 + 9x + \frac{81}{4}$, so $a = 9$ and $b = \frac{81}{4}$, and $a + b = \boxed{\dfrac{117}{4}}$.

5. Milvia receives the following question on her math test: "The area of a 30° sector of a circle with radius r is equal to $\frac{p}{q}\pi$ for relatively prime positive integers p and q, where q may be 1. Compute $p + q$." Given that she correctly answers 28, compute the sum of all possible positive integer values of r.

   (a) 25
   (b) 28
   (c) 30
   (d) 37
   (e) none of the above

   Answer: $\boxed{B}$

   The area of the sector is $\pi \cdot \frac{r^2}{12}$; to cut down on brute-forcing, we notice that either $\frac{p}{q} > 3$ (for example) or $\frac{p}{q} = \frac{1}{12}, \frac{1}{3}, \frac{3}{4}, \frac{4}{3}, \frac{25}{12}, \frac{3}{1}$ (none of which have $p + q = 28$). Since $\frac{21}{7} = 3$, it suffices to check $\frac{22}{6}$, $\frac{23}{5}$, $\frac{24}{4}$, $\frac{25}{3}$, $\frac{26}{2}$, and $\frac{27}{1}$ as possibilities. We find that only $\frac{25}{3}$ and $\frac{27}{1}$ work (from $r = 10, 18$ respectively), so the answer is $10 + 18 = \boxed{28}$.

6. How many ordered tuples (a, b, c) of positive integers satisfy $a+b+c<10$?
   (a) 45
   (b) 56
   (c) 70
   (d) 84
   (e) none of the above

   Answer: $\boxed{D}$

   Consider the inequality $a+b+c=n$ for some positive integer $3 \leq n \leq 9$. By stars-and-bars, there are $\binom{n-1}{2}$ solutions. Summing $\binom{n-1}{2}$ from $n=3$ to $n=9$ yields $\binom{9}{3}=\boxed{84}$ by the hockey-stick identity.

7. Triangle ABC has $AB=13$, $BC=14$, and $CA=15$. Point D lies on $\overline{AB}$ with $AD=5$, and point E lies on $\overline{CA}$ with $CE=10$. The area of $\triangle ADE$ can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m+n$.

   Answer: $\boxed{153}$

   This is $\frac{5}{13} \cdot \frac{5}{15}=\frac{5}{39}$ times the area of $\triangle ABC$, which is 84 by Heron's formula; or $\frac{140}{13}$, which gives $m+n=140+13=\boxed{153}$.

8. For each positive integer n, let $s(n)$ be the sum of the digits of n. Compute the sum of all positive integers $n \leq 100$ such that $s(n^2)=s(n)^2$.

   Answer: $\boxed{276}$

   For $n \leq 100$, we have $n^2 \leq 10000$, and so $s(n^2) \leq s(9999)=36$, implying that $s(n) \leq 6$. If $s(n)=6$, then $s(n^2)=36$, which is not possible to satisfy with $s(n)=6$. If $s(n)=5$, then $s(n^2)=25$, but once again, one may check that no values of n satisfy both equations simultaneously. For $s(n)=4$, we have $s(n^2)=16$, which happens when $n=31$, $n=22$, and $n=13$. For $s(n)=3$, and $s(n^2)=9$, we have $n=30$, $n=21$, $n=12$, and $n=3$. For $s(n)=2$ and $s(n^2)=4$, we get $n=20$, $n=11$, and $n=2$. For $s(n)=s(n^2)=1$, we have only $n=100$, $n=10$, and $n=1$. The sum of the values of n satisfying $s(n^2)=s(n)^2$ is then $31+22+13+30+21+12+3+20+11+2+100+10+1=\boxed{276}$.

9. Compute the number of denominators of all fractions between $\frac{92}{99}$ and $\frac{93}{100}$ that have denominators less than or equal to 2022.

   Answer: $\boxed{190}$

   As $100 \cdot 20=2022<2022$, but $99 \cdot 21=2079>2022$, the denominators can be any positive integer between $99k$ and $100k$ for $1 \leq k \leq 20$. This gives $0+1+2+\cdots+19=\boxed{190}$ fractions.

10. A mixture is 19 parts Substance A to 11 parts Substance B. If another mixture which consists of 62 percent Substance A to 38 percent Substance B is mixed well with the first mixture, in order for the new mixture to be at least $\frac{5}{8}$ Substance A by volume, the second mixture must have had total volume at most t times that of the first mixture. Compute t. Express your answer as a common fraction.

    Answer: $\boxed{\frac{5}{3}}$

    Call the mixtures M_1 and M_2 respectively. For a new mixture that is 1 part M_1 and t parts M_2, its concentration of Substance A is $\frac{1 \cdot \frac{19}{30}+t \cdot \frac{31}{50}}{t+1}$. For this to be at least $\frac{5}{8}$, we must have $\frac{76}{15}+\frac{124t}{25} \geq 5t+5$, or $\frac{1}{15} \geq \frac{1}{25}t$. Then $t \leq \boxed{\frac{5}{3}}$.

11. A rectangle has area $\frac{144}{25}$ and integer perimeter, and side lengths that, when multiplied by 25, are integers. Compute the sum of all possible values for its longer side length.

    Answer: $\boxed{44}$

    The side lengths are either of the form $\frac{a}{25}$ (with $\gcd(a, 25) = 1$) and b, or $\frac{a}{5}$ and $\frac{b}{5}$ for integers a and b with $\gcd(a, 5) = \gcd(b, 5) = 1$. Only the latter case can give an integer perimeter (with $a + b$ being a multiple of 5), so $ab = 144$, and $(a, b) = (12, 12), (16, 9), (18, 8), (24, 6), (36, 4), (48, 3), (72, 2), (144, 1)$. Among these, $(a, b) = (16, 9), (24, 6), (36, 4), (144, 1)$ all have $a + b$ a multiple of 5, and the sum of the possible values of $\frac{a}{5}$ is $\frac{16+24+36+144}{5} = \boxed{44}$.

12. Lexine wants to buy exactly 20 donuts at MoreDonuts. The available flavors are vanilla, strawberry, chocolate cream, and jelly. Among the pairs {vanilla, strawberry} and {chocolate cream, jelly} of different donut types, Lexine wants to buy exactly 10 donuts of either of those types (so the numbers of vanilla and strawberry donuts she buys must sum up to 10, and likewise for the other pairs). In how many ways can she buy donuts? (Two ways are considered different if the number of donuts of any given type is different.)

    Answer: $\boxed{121}$

    If Lexine buys $0 \le v \le 10$ vanilla donuts, she must buy $10 - v$ strawberry donuts, and if she buys $0 \le c \le 10$ chocolate cream donuts, she must buy $10-c$ jelly donuts. Each choice of v and c determines $10 - v$ and $10 - c$ respectively, so Lexine has $11^2 = \boxed{121}$ ways to buy donuts.

13. Palmer starts at one of the squares of a square grid with 2 rows and 3 columns. He travels to an adjacent square that he has not already visited, and wants to eventually reach every square in the grid. In how many ways can he do this?

    Answer: $\boxed{16}$

    If Palmer starts at one of the four corners of the grid (without loss of generality, the top-left one), he can take the paths DRRUL, DRURD, or RRULL. By symmetry, there are a total of $3 \cdot 4 = 12$ corner paths. If Palmer starts from a center square (suppose the top one), he has paths LDRRU or RDLLU, for another $2 \cdot 2 = 4$ paths. Altogether, there are $12 + 4 = \boxed{16}$ paths covering every square on the grid, depending on starting point.

14. Suppose that, for a prime number p, $20p$ has n positive integer divisors. Compute the sum of the possible values of n.

    Answer: $\boxed{29}$

    Since $20 = 2^2 \cdot 5$, we can do casework depending on whether $p = 2$, $p = 5$, or $p \ne 2, 5$. If $p = 2$, we get $20p = 40 = 2^3 \cdot 5$, which has 8 divisors. If $p = 5$, we get $20p = 60 = 2^2 \cdot 5^2$, which has 9 divisors. Otherwise, we get $2^2 \cdot 5 \cdot p$, which has 12 divisors. The sum of the possible numbers of divisors is $8 + 9 + 12 = \boxed{29}$.

15. If the real roots of $x^3 - 30x^2 - kx + 20k$ are in geometric progression for some real number k, compute k. Express your answer in simplest radical form.

    Answer: $\boxed{-300\sqrt{6}}$

    Let the three roots be a, ar, and ar^2 for some real numbers a and r. By Vieta's formulas, they sum to 30, so $a(1+r+r^2) = 30$, and also $a(ar)+a(ar^2)+ar(ar^2) = a^2(r+r^2+r^3) = -k$; furthermore, $a(ar)(ar^2) = a^3r^3 = -20k$, so that $\frac{a^3r^3}{a^2(r+r^2+r^3)} = a \cdot \frac{r^2}{1+r+r^2} = 20$. This gives $a^2r^2 = 600$ upon multiplying with $a(1+r+r^2) = 30$, meaning that $a^3r^3 = 600\sqrt{600} = -20k$, and $k = -30\sqrt{600} = \boxed{-300\sqrt{6}}$.

16. Beginning with a 3×3 grid of squares, Rebekah selects one of the squares in each column, each with equal probability. She then crosses it, and all squares below it in its column, out with an X. She then repeats this process with any remaining squares in each column, until all squares in the grid are crossed out. Compute the expected number of squares she selects. Express your answer as a common fraction.

    Answer: $\boxed{\frac{11}{2}}$

    Consider each column separately. If the bottom square is crossed out with probability $\frac{1}{3}$, then with probability $\frac{1}{2}$, it will take 2 more crosses to fill that column, and with probability $\frac{1}{2}$, only one cross of the top square. Thus, the expected number of Xs in this case is $\frac{5}{2}$. With probability $\frac{1}{3}$, the middle square is crossed out, and it will certainly take exactly 2 crosses to fill the column. Finally, with probability $\frac{1}{3}$, the top square is crossed, and we are done immediately. It follows that the expected number of Xs needed per column is $\frac{1}{3}\left(\frac{5}{2}+2+1\right) = \frac{11}{6}$. By the linearity of expectation, the total number of squares crossed out is $3 \cdot \frac{11}{6} = \boxed{\frac{11}{2}}$.

17. How many positive integers have base-4 representations with at most 5 digits that contain at least three of the same digit in a row?

    Answer: $\boxed{144}$

    We employ complementary counting to subtract from $4^5 - 1 = 1023$ the total of such integers that do *not* contain three of the same digit in a row. First consider those integers with exactly 5 digits in base 4. If the second digit is equal to the first digit (which can be chosen in 3 ways), the third digit can be one of three possibilities, from which there are $4^2 - 1 = 15$ possibilities for digits 4 and 5 (from the left to the right). Otherwise, if the second digit is not equal to the first digit (with $3 \cdot 3 = 9$ choices for these digits), and the third digit is equal to the second digit, we have $3 \cdot 4 = 12$ choices for the last two digits; if the third and second digits differ (with 3 choices for the third digit), we have $4^2 - 1 = 15$ choices for digits 4 and 5. This gives a total of $3 \cdot 3 \cdot 15 + 9 \cdot 12 + 9 \cdot 3 \cdot 15 = 648$ numbers with *exactly five* digits that don't have three consecutive identical digits.

    For four-digit numbers in base 4, the process is similar: we have 3 choices for the first digit. If the second digit is identical, then the third digit can be anything but the first/second digit, giving 3 choices, and the fourth digit is unrestricted, which yields 12 numbers. If, on the other hand, the first and second digits are unequal (in 3 ways), there are $4^2 - 1 = 15$ choices for digits 3 and 4. This case gives $3(12 + 3 \cdot 15) = 171$ additional numbers.

    For three-digit numbers, we have $3 \cdot 4^2 - 3 = 45$ more numbers (excluding only 111, 222, and 333 in base 4); we also have $3 \cdot 4 = 12$ base-4 two-digit numbers and 3 base-4 single-digit numbers. Altogether, we obtain $648 + 171 + 45 + 12 + 3 = 879$ numbers that do not have three of the same digit in a row, or $1023 - 879 = \boxed{144}$ that do.

    Remark. Just to be sure (and for fun), I confirmed this with a Python program! The program basically took all integers from 1 to 1023 inclusive, converted them to their corresponding base-4 strings (not including leading zeros) with five or fewer digits, and checked for three instances of the same digit in a row by implementing a counter that went up by 1 every time there were two consecutive digits, reset to zero when the streak was broken by two consecutive non-identical digits, and checking to see if that counter ever reached 2. Ironically enough, this took me about 2 hours to write, whereas this solution took much less time (albeit still a nontrivial amount, considering the casework involved).

18. The side lengths of a triangle with area 1 are 2, s, and t for some real numbers s and t with $s + t = 3$. Compute st. Express your answer as a common fraction.

Answer: $\boxed{\frac{41}{20}}$

The triangle's semi-perimeter is $\frac{2+3}{2} = \frac{5}{2}$. By Heron's formula, its area, which is equal to 1, can be written as $\sqrt{\frac{5}{2}\left(\frac{5}{2}-2\right)\left(\frac{5}{2}-s\right)\left(\frac{5}{2}-t\right)}$, and so $\left(\frac{5}{2}-s\right)\left(\frac{5}{2}-t\right) = \frac{4}{5}$. Expanding, we obtain $\frac{25}{4}-\frac{5}{2}t-\frac{5}{2}s+st = \frac{4}{5}$, and since $s+t=3$, this simplifies to $st = \boxed{\frac{41}{20}}$.

19. Triangle ABC has $AB = 13$, $BC = 14$, and $CA = 15$. Let O be the circumcenter of $\triangle ABC$ and let $P \in \overline{BC}$ be the foot of the A-altitude of $\triangle ABC$. Line $\overline{OP}$ intersects $\overline{AC}$ at point D. Compute $\frac{AD}{DC}$.

Answer: $\boxed{\frac{64}{99}}$

Without loss of generality, assign the coordinates $A = (5, 12)$, $B = (0, 0)$, and $C = (14, 0)$. The circumradius of $\triangle ABC$ satisfies the equation $\frac{abc}{4R} = [\triangle ABC]$ (where $[\triangle ABC]$ is the area of the triangle, which is 84 by Heron's formula, and a, b, c are its side lengths). Hence, $\frac{13\cdot 14\cdot 15}{4R} = 84$, and $R = \frac{65}{8}$. If we recall that the circumcenter of $\triangle ABC$ is the intersection of its perpendicular bisectors, we know that its x-coordinate must be 7, which saves us brute-forcing of another perpendicular bisector – immediately we know the y-coordinate must be $\frac{33}{8}$, and $O = \left(7, \frac{33}{8}\right)$.

The coordinates of P are $(5, 0)$, so $\overline{OP}$ has equation $y = \frac{33}{16}x - \frac{165}{16}$. This intersects the line segment $\overline{AC}$, whose equation is $y = -\frac{4}{3}x + \frac{56}{3}$ (given that it passes through the points $(5, 12)$ and $(14, 0)$), where $y = \frac{33}{16}x - \frac{165}{16} = -\frac{4}{3}x + \frac{56}{3}$, or, upon multiplying through by 48, $48y = 99x - 495 = -64x + 896$. Solving for x gives $x = \frac{1391}{163}$, which is enough to determine $\frac{AD}{DC}$. Indeed, this is $\frac{\frac{1391}{163}-5}{14-\frac{1391}{163}} = \boxed{\frac{64}{99}}$.

20. Adele draws a rhombus in the coordinate plane with vertices at the points $(0, \pm 12)$ and $(\pm 5, 0)$. She then draws a circle centered at the origin with radius 5. The area of the quadrilateral whose vertices are the points of intersection of the rhombus and the circle (other than $(\pm 5, 0)$) can be written in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. Find the sum of all prime numbers that divide either p or q.

Answer: $\boxed{47}$

The circle's equation is $x^2 + y^2 = 25$, while the sides of the rhombus have equations $y = -\frac{12}{5}x + 12$ (from $(0, 12)$ to $(5, 0)$), $y = \frac{12}{5}x - 12$ (from $(5, 0)$ to $(0, -12)$), $y = -\frac{12}{5}x - 12$ (from $(0, -12)$ to $(-5, 0)$), and $y = \frac{12}{5}x + 12$ (from $(-5, 0)$ to $(0, 12)$). By symmetry, we need only calculate one of the points of intersection, and then flip the signs of its coordinates to get the other three intersection points. Hence, if the coordinates of, say, the intersection point in the first quadrant are (x, y), the area of the quadrilateral (indeed, a rectangle) whose vertices are the intersection points will be $4xy$.

We now solve the system of equations

$$x^2 + y^2 = 25, y = -\frac{12}{5}x + 12.$$

Directly substituting the second equation into the first gives $\frac{169}{25}x^2 - \frac{288}{5}x + 119 = 0$, or $169x^2 - 1440x + 2975 = 0$, from which we apply the quadratic formula to get that

$$x = \frac{1440 \pm \sqrt{1440^2 - 26^2 \cdot 2975}}{338} = \frac{1440 \pm 250}{338} = \frac{595}{169}, 5.$$

Since we are looking for the "non-trivial" intersection points (those not equal to $(0, \pm 5)$ or $(\pm 5, 0)$), we take $x = \frac{595}{169}$, from which $y = \frac{600}{169}$, and thus $4xy = \frac{4\cdot 595\cdot 600}{169^2} = \frac{2^5\cdot 3\cdot 5^2\cdot 7\cdot 17}{13^4}$. Finally, we have $p = 2^5\cdot 3\cdot 5^2\cdot 7\cdot 17$ and $q = 13^4$, and the sum of the prime factors of either p or q is $2+3+5+7+13+17 = \boxed{47}$.