cyber math

Online Open Tournament

Contest Solutions (Europe/Africa)

High School Division

Saturday, March 26, 2022

The problems and solutions for this competition were prepared by the teaching staff of CyberMath Academy. Thanks to Thinula de Silva and Freya Edholm for reviewing these problems and solutions!

1. For how many positive integers n will repeatedly subtracting n from 1000 eventually produce 1?

   (a) 4
   (b) 6
   (c) 8
   (d) 9
   (e) none of the above

   Answer: $\boxed{C}$

   We want n to be a factor of $999 = 3^3 \cdot 37$, which has $(3+1)(1+1) = \boxed{8}$ factors.

2. A rectangle has side lengths in the ratio 3:4. If the diagonal has length d, and the rectangle's area is 1, compute d. Express your answer as a common fraction in simplest radical form.

   Answer: $\boxed{\dfrac{5\sqrt{3}}{6}}$

   Note that $\left(\frac{3}{5}, \frac{4}{5}, 1\right)$ is a Pythagorean triple, so the rectangle's side lengths are $\frac{3}{5}d$ and $\frac{4}{5}d$. The rectangle's area is 1, so $\frac{12}{25}d^2 = 1$, and $d^2 = \frac{25}{12}$, meaning that $d = \boxed{\dfrac{5\sqrt{3}}{6}}$.

3. A cafe serves milk in five flavors: plain, strawberry, chocolate, vanilla, and caramel. Mikhail wants to drink milk at the cafe for five consecutive days, but does not want to drink the same flavor for two days in a row, nor does he want to drink plain milk on the last day or chocolate milk on the first day. In how many ways can he decide which milk flavors to drink on each of the five days?

   (a) 625
   (b) 768
   (c) 1024
   (d) 1278
   (e) none of the above

   Answer: $\boxed{E}$

   We apply the inclusion-exclusion principle, subtracting from the total of $5 \cdot 4^4 = 1280$ potential patterns the total that include plain milk on day 5 or chocolate milk on day 1, and then add back the over-subtracted ones that have both. If Mikhail were to drink plain milk on day 5, backtracking, he would have 4 choices for the milk he drinks on day 4, 4 choices for day 3, 4 choices for day 2, and 4 choices for day 1, giving $4^4 = 256$ bad patterns; likewise for chocolate milk on day 1.

   A pattern that has both plain milk on day 5 and chocolate milk on day 1 is of the form HXXXP, where X is a placeholder for an arbitrary milk flavor. Consider the possible combos of day 2/4 milk flavors. There are $4^2 = 16$ such combos, 3 of which consist of the same milk flavor (SS, VV, or RR) (note: here H is chosen to stand for cHocolate, to avoid confusion with caRamel). For each of the 13 combos with different flavors on days 2 and 4, Mikhail has 3 choices for the day 3 flavor, and for each of the 3 combos with the same flavor on days 2 and 4, Mikhail can choose the day 3 milk flavor in 4 ways. This gives a total of $13 \cdot 3 + 3 \cdot 4 = 51$ bad combos that are overcounted/were subtracted twice. Finally, this gives $1280 - 256 - 256 + 51 = \boxed{819}$ "good" patterns.

   Alternatively, the following table (row = second day milk flavor, column = fourth day milk flavor) gives the numbers of possibilities for each combo of second/fourth day milk flavors:

	P	S	H	V	R
P	48	27	27	27	27
S	36	36	27	27	27
H	48	36	48	36	36
V	36	27	36	27	27
R	36	27	27	27	36

which sum to $\boxed{819}$. (For P/P for example, the pattern is XPXPX, which gives 3 choices for day 1 (not plain or chocolate), 4 choices for day 3 (anything but plain), and likewise 4 choices for day 5. With a little bit of work, this can be extrapolated to the other 24 combos.)

4. Compute the sum of the coordinates of all rational points (points with rational coordinates) that lie on the circle $(x-8)^2+(y-6)^2=100$.

   (a) 168
   (b) 140
   (c) 112
   (d) 56
   (e) none of the above

   Answer: $\boxed{A}$

   It suffices to count the number of rational points and multiply by $8+6=14$, since each is a translation of the corresponding rational point on $x^2+y^2=100$ by $(8,6)$. The only nontrivial integer solution (x,y) to $x^2+y^2=100$ is $(6,8)$ and permutations up to sign, giving $2!\cdot 2^2=8$ such rational points. There are also 4 rational points of the form $(\pm 10,0)$ and $(0,\pm 10)$, so in total, there are 12 rational points, and the sum of the coordinates of the rational points lying on the translated circle is $\boxed{168}$.

5. At least 4 percent of the positive integers between n and $2n$, inclusive, are perfect squares. Compute the largest possible value of the positive integer n.

   Answer: $\boxed{121}$

   We certainly require that $\sqrt{n}(\sqrt{2}-1)+1 \geq \frac{n+1}{25}$, in the case that n or $2n$ are a perfect square (note that both cannot be at the same time). By inspection, a reasonable approximation seems to be $n=100$, which gives approximately 4.142 on the LHS and 4.04 on the RHS. Can we do better? We can check that there are 5 perfect squares in $[100,200]$, and that the interval contains 101 integers. If we can maintain 5 perfect squares while not going above 125 integers, we will have a larger upper bound on n. (Note that we can't have 6 or more perfect squares in $[n,2n]$, since for $n\geq 146$, which is the point at which $\sqrt{n}(\sqrt{2}-1)\geq 5$ for the first time, we have that $\sqrt{n}(\sqrt{2}-1)$ increases by 1 (from 4 to 5) for an increase of n by 46, which is certainly much larger than 25! This differential only increases as n grows larger.) For $101\leq n\leq 112$, we will only have 4 perfect squares in $[n,2n]$, since the square 100 disappears and no perfect square replaces it (the next above 200 is 225). From $n=113$ up to $n=121$, we once again have 5 perfect squares in $[n,2n]$, and it disappears at $n=122$. The next perfect square above 225 is $256=128\cdot 2$, but this contradicts $n\leq 125$. So $n=\boxed{121}$ is the best we can do.

6. Suppose that the real part of $z=(x+iy)^3$ is 259 for positive integers x and y, where $i=\sqrt{-1}$. Compute the imaginary part of z.

   Answer: $\boxed{286}$

   Expanding (by the binomial theorem) and equating coefficients, we obtain $z=x^3+3x^2iy-3xy^2-iy^3 \implies x^3-3xy^2=x(x^2-3y^2)=259$. For positive integers x and y, as $259=37\cdot 7$, this implies that $x=7$ and $y=2$, so the imaginary part of z, $3x^2y-y^3$, evaluates to $\boxed{286}$.

7. A base-b positive integer N whose base-10 value is 83 is written on a blackboard. When its rightmost digit is erased, its base-10 value becomes 16. When N is read as a base-10 number, what is $N + b$?

Answer: $\boxed{318}$

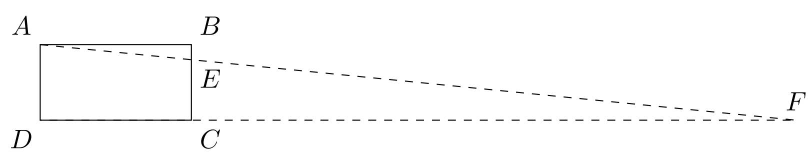
Suppose that N has two digits in base b. That erasing its rightmost digit produces 16_{10} implies that the remaining digit has a value of $(16)_{10}$, but then $b \geq 17$, which suggests that $N \geq 16 \cdot 17 = 272$, a contradiction.

If, instead, we assume that N has three digits, erasing the rightmost digit to produce 16_{10} implies that $16b + d = 83$ for some ordered pair (b, d) with $2 \leq b > d$. This forces $(b, d) = (5, 3)$, hence $N = 313_5$ and $N + b = 313 + 5 = \boxed{318}$. (It is easy to show that N cannot have four digits or more: this would imply $b^3 \leq 83$, or $b \leq 4$, and we check that $b = 2, 3, 4$ all fail to satisfy the second condition in the problem statement.)

8. Rectangle $ABCD$ has $AB = 2$ and $BC = 1$. Point E lies on $\overline{BC}$ such that the extension of $\overline{AE}$ past E intersects the extension of $\overline{CD}$ at F and the area of $\triangle EFD$ is 2022. Compute BE. Express your answer as a common fraction.

Answer: $\boxed{\dfrac{1}{2023}}$

We refer to the following diagram (not to scale).



Let $EC = h$; by similarity, $\frac{FC}{CE} = \frac{FD}{DA}$, so that $\frac{FC}{h} = FC + 2$, or $h = 1 - \frac{2}{FC+2}$. Given that the area of $\triangle EFD$ is 2022, we have $h \cdot FD = 4044$, or $h \cdot (FC + 2) = 4044$. Substituting $u = 1 - \frac{2}{FC+2}$, we get $(FC + 2) - 2 = 4044$, or $FC = 4044$, and thus, $h = \frac{2022}{2023}$. Finally, $BE = 1 - h = \boxed{\dfrac{1}{2023}}$.

9. Three fair six-sided dice, each labeled with the integers from 1 through 6 inclusive, are rolled. Compute the probability that no pair of dice rolls has a product that is a perfect square, but the product of all three dice rolls is a perfect square. Express your answer as a common fraction.

Answer: $\boxed{\dfrac{1}{36}}$

The possible products of all three dice are 144, 100, 64, 36, 25, 16, 9, 4, and 1. If the product is 1 or the square of a prime p, it is impossible to satisfy the given condition, since the rolls must be p, p, and 1 (or, if $p = 2$, the permutations of $(1, 1, 4)$, which again are not valid) and $p \cdot p = p^2$. Thus, the only possible products that remain are 144, 100, 64, 36, and 16.

We can express 144 only as the product of 6, 6, and 4, but because $6 \cdot 6 = 36$, we once again rule this out. We similarly can write 100 only as $5 \cdot 5 \cdot 4$, which does not count because of the two 5s. We can only write $64 = 4 \cdot 4 \cdot 4$ (which is again not a valid combination), because 64 is a power of 2, and 8 is not a possible die roll. We can write $36 = 6 \cdot 3 \cdot 2$, which *does* satisfy the given condition, hence contributing $3! = 6$ possible roll sequences. We can also write 36 as $6 \cdot 6 \cdot 1$ or as $4 \cdot 3 \cdot 3$, but these are not valid roll combinations because they contain two of the same roll; the same goes for 16, which can only be written as $4 \cdot 4 \cdot 1$ (again, 8 is not a possible roll). It follows that the permutations of $(6, 3, 2)$ are the only valid roll combinations, for a probability of $\frac{6}{6^3} = \boxed{\dfrac{1}{36}}$.

10. For each positive integer n, let $Z(n)$ be the set of all positions (from left to right) of the instances of the digit 0 in n. For example, $Z(2022) = \{2\}$, and $Z(102030405) = \{2, 4, 6, 8\}$. Compute

$$\sum_{n=1}^{2022} \sum_{d \in Z(n)} d.$$

Answer: $\boxed{1213}$

Among the single-digit positive integers, there are no zeros. Among the two-digit integers, there is only a zero as the units digit for the multiples of 10, and a zero is never the tens digit. For each of these integers n, $Z(n) = \{2\}$, so the contribution to the sum is $2 \cdot 9 = 18$.

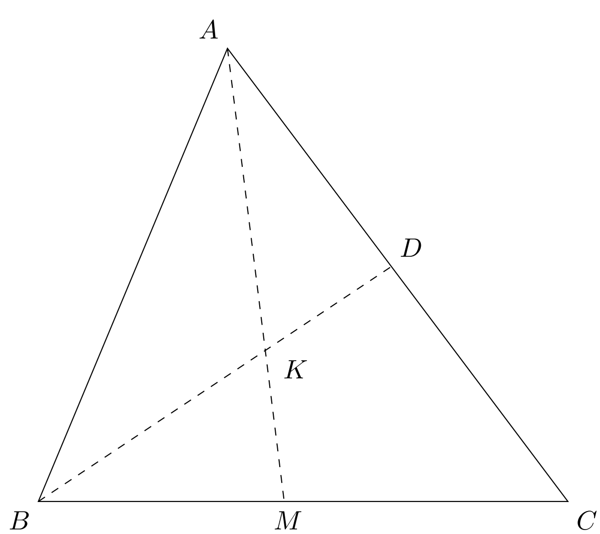
For three-digit integers, there are 9 integers (the multiples of 100) with two 0s. For these integers n, the sum of the elements of $Z(n)$ is $2 + 3 = 5$. Those with one zero are of the form $a0b$ for $1 \le a, b \le 9$, with 81 such integers, or the form $ab0$ for $1 \le a, b \le 9$ (where there are again 81 such integers, and no integer is double-counted). In the former case, the sum of the d's in the $Z(n)$'s is $2 \cdot 81 = 162$; in the latter, it is $3 \cdot 81 = 243$.

For four-digit integers from 1000 to 1099, the sum of the d's is equal to $((2 + 3 + 4) + 9 \cdot (2 + 3)) + 9 \cdot ((2 + 4) + 9 \cdot 2) = (9 + 45) + 9 \cdot (6 + 18) = 54 + 9 \cdot 27 = 297$ (varying the tens digit from 0 to 9). In the range 1100-1999, the sum of the d's is identical to the sum for 100-999. For 2000 through 2022, we can compute manually: the sum for 2000 is $2 + 3 + 4 = 9$; the sums for 2001-2009 are all $2 + 3 = 5$; the sum for 2010 is $2 + 4 = 6$; the sums for 2011-2019 are all 2; the sum for 2020 is $2 + 4 = 6$; and the sums for 2021 and 2022 are both 2. In total, we get a sum of $18 + 297 + 2(162 + 243) + 9 + 9 \cdot 5 + 6 + 9 \cdot 2 + 6 + 2 + 2 = 18 + 297 + 810 + 88 = \boxed{1213}$.

11. Triangle ABC has $AB = 13$, $BC = 14$, and $CA = 15$, with M lying on $\overline{BC}$ so that $\overline{AM}$ bisects $m\angle BAC$. Point D lies on $\overline{AC}$ so that $\overline{BD}$ and $\overline{AM}$ intersect at point K. If the distance from K to $\overline{BC}$ is 4, then $\frac{AD}{DC} = \frac{m}{n}$ for relatively prime positive integers m and n. Compute $m + n$.

Answer: $\boxed{27}$

We draw the following diagram:



Note that K is the incenter of $\triangle ABC$, since $\overline{AM}$ is an angle bisector and the inradius of $\triangle ABC$ is 4 (from the formula $[\triangle ABC] = rs$, where $[\triangle ABC]$ denotes the area of $\triangle ABC$, in this case 84 from

Heron's formula, r is the inradius, and s is the semi-perimeter, in this case 21). As the incenter of a triangle is the intersection of its internal angle bisectors, $\overline{BD}$ is the internal angle bisector of $m\angle ABC$, meaning that $\frac{AD}{DC} = \frac{AB}{BC} = \frac{13}{14}$ by the angle bisector theorem. Hence, $m + n = 13 + 14 = \boxed{27}$.

12. Let x and y be distinct real numbers such that $x^2 + 7x + 11 = y^2 + 7y + 11 = 0$. Compute

$$\frac{x^2}{x^2y + y^3} + \frac{y^2}{x^3 + xy^2}.$$

Answer: $\boxed{-\dfrac{112}{297}}$

The desired quantity can be written as

$$\frac{x^2}{y(x^2+y^2)} + \frac{y^2}{x(x^2+y^2)} = \frac{1}{x^2+y^2}\left(\frac{x^2}{y} + \frac{y^2}{x}\right) = \frac{1}{x^2+y^2} \cdot \frac{x^3+y^3}{xy} = \frac{x^3+y^3}{x^3y + xy^3}.$$

By Vieta's formulas,

$$x^3 + y^3 = (x+y)^3 - 3(x^2y + xy^2) = (x+y)^3 - 3xy(x+y) = (-7)^3 - 3(11)(-7) = -112,$$

and

$$x^3y + xy^3 = xy(x^2+y^2) = xy((x+y)^2 - 2xy) = 11((-7)^2 - 2 \cdot 11) = 297,$$

so $\frac{x^2}{x^2y+y^3} + \frac{y^2}{x^3+xy^2} = \boxed{-\dfrac{112}{297}}$.

13. Delilah wants to buy exactly 18 donuts at MoreDonuts. The available flavors are vanilla, strawberry, and chocolate cream, but station 1 only serves vanilla and strawberry, station 2 only serves vanilla and chocolate cream, and station 3 only serves strawberry and chocolate cream. Suppose Delilah buys exactly 6 donuts from each station, and no more than 12 donuts of any flavor. In how many ways can she buy donuts? (Two ways are different if there are a different number of any given type of donut.)

Answer: $\boxed{127}$

By stars and bars, Delilah has $\binom{18+3-1}{3-1} = \binom{20}{2}$ choices for donuts with no restrictions; however, she may not buy more than 12 donuts of any kind, since each flavor is only available from 2 stations. (All combinations of donuts with 12 or fewer donuts of one kind are achievable; i.e. if she buys v vanilla, s strawberry, and c chocolate cream such that (without loss of generality) $v \geq s \geq c$ and $v, s, c \leq 12$, she can buy 6 vanilla donuts from station 1, $v - 6$ vanilla donuts and $12 - v$ strawberry donuts from station 2, and $s + v - 12$ strawberry cream donuts and c chocolate donuts from station 3. As $v \geq 6$ is forced from it being the largest of the three, and $s + v \geq 12$ (from $c \leq 6$, which is always true when $v \geq s \geq c$), this is always possible.) Hence, we count the number of cases with 13 or more vanilla donuts, and triple to account for the analogous cases with 13 or more strawberry and 13 or more chocolate cream donuts. There are $6 + 5 + 4 + 3 + 2 + 1 = 21$ ways for Delilah to buy at least 13 vanilla donuts (if she buys 13, she must buy 5 other donuts among strawberry and chocolate cream, and the possible combinations here are $5 + 0$, $4 + 1$, $3 + 2$, $2 + 3$, $1 + 4$, and $0 + 5$; likewise for other cases), so in total, there are $21 \cdot 3 = 63$ bad cases. Altogether, Delilah has $\binom{20}{2} - 63 = \boxed{127}$ ways to buy donuts.

14. For each positive integer n, let $R(n)$ be the ratio of the number of even divisors of n to the number of divisors of n. Compute the fractional part of $R(1) + R(2) + R(3) + \cdots + R(100)$.

Answer: $\boxed{\dfrac{62}{105}}$

Factorizing n as $2^k \cdot m$ for odd m, we have $R(n) = \frac{k}{k+1}$, since we can choose for an even divisor of n any positive exponent of 2, and for an odd divisor of n, the exponent of 2 must be 0. There are 50 odd numbers n from 1 to 100, for which $R(n) = 0$; 25 numbers n for which the largest power of 2 dividing n is 2^1, hence $R(n) = \frac{1}{2}$; 13 numbers n for which the largest power of 2 dividing n is 2^2, so that $R(n) = \frac{2}{3}$; 6 numbers n for which $R(n) = \frac{3}{4}$; 3 numbers n for which $R(n) = \frac{4}{5}$; 2 numbers n such that $R(n) = \frac{5}{6}$; and 1 number n such that $R(n) = \frac{6}{7}$. Altogether, the sum evaluates to $25 \cdot \frac{1}{2} + 13 \cdot \frac{2}{3} + 6 \cdot \frac{3}{4} + 3 \cdot \frac{4}{5} + 2 \cdot \frac{5}{6} + 1 \cdot \frac{6}{7}$, whose fractional part is that of $\frac{1}{2} + \frac{2}{3} + \frac{1}{2} + \frac{2}{5} + \frac{2}{3} + \frac{6}{7}$, or that of $\frac{1}{3} + \frac{2}{5} + \frac{6}{7} = \frac{167}{105}$, which is $\boxed{\frac{62}{105}}$.

15. For each positive integer $n = p_1^{j_1} p_2^{j_2} p_3^{j_3} \cdots p_k^{j_k}$, where the p_i are distinct prime numbers and the j_i are positive integers, define

$$f(n) := \prod_{i=1}^{k} (p_i - 1).$$

Let $\varphi(n)$ be the number of positive integers less than or equal to n that are relatively prime to n. Compute the largest positive integer N such that

$$\sum_{n=1}^{N} \frac{f(n)}{\varphi(n)} < 10.$$

Answer: $\boxed{12}$

Recall that, where $n = p_1^{j_1} p_2^{j_2} p_3^{j_3} \cdots p_k^{j_k}$,

$$\phi(n) = n \cdot \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right).$$

In particular, whenever we have a prime factor power of the form $p_i^{j_i}$ for $j_i \geq 2$, this contributes a multiplicative factor of $\frac{1}{p_i^{j_i - 1}}$ to $\frac{f(n)}{\varphi(n)}$. For instance, $\frac{f(60)}{\varphi(60)} = \frac{8}{8 \cdot 2} = \frac{1}{2}$, while $\frac{f(120)}{\varphi(120)} = \frac{8}{8 \cdot 4} = \frac{1}{4}$.

Up to 10, the only positive integers divisible by a power of a prime are 4, 8, and 9, so that the sum of $\frac{f(n)}{\varphi(n)}$ up to 10 is $7 + \frac{1}{2} + \frac{1}{4} + \frac{1}{3}$. Then $\frac{f(11)}{\varphi(11)} = 1$, $\frac{f(12)}{\varphi(12)} = \frac{1}{2}$, $\frac{f(13)}{\varphi(13)} = 1$, and our sum up to this point is greater than 10 for the first time; hence $N \leq \boxed{12}$.

16. Compute the number of ordered pairs (a, b) of positive integers with $1 \leq a, b \leq 100$ such that the system of equations $x - y = a$, $x^3 - y^3 = b$ has a real solution (x, y).

Answer: $\boxed{509}$

Recall that $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$; we can write $x^2 + xy + y^2 = (x - y)^2 + 3xy = \frac{b}{a}$. As $x - y = a$, this implies that $xy = \frac{b}{3a} - \frac{a^2}{3}$. From $y = x - a$, we get that $x^2 - ax = \frac{b}{3a} - \frac{a^2}{3}$, which simplifies to $3ax^2 - 3a^2x + (a^3 - b) = 0$. From here we get that $x = \frac{3a^2 \pm \sqrt{12ab - 3a^4}}{6a}$, so that $12ab \geq 3a^4$, or $4b \geq a^3$. By casework on a, we get 100 ordered pairs for $a = 1$, 99 pairs for $a = 2$, 94 for $a = 3$, 85 for $a = 4$, 69 for $a = 5$, 47 for $a = 6$, 15 for $a = 7$, and none for $a \geq 8$, giving a total of $\boxed{509}$ ordered pairs (a, b).

17. Triangle ABC has $AB = 13$, $BC = 14$, and $CA = 15$. Point P lies on $\overline{BC}$ such that the circumcircle Ω of $\triangle APC$ intersects $\overline{AB}$ at $Q \neq A$ and $AQ = \frac{13}{4}$. Then $BP = \frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

Answer: $\boxed{563}$

Let $A = (5, 12)$, $B = (0, 0)$, and $C = (14, 0)$, and let $P = (p, 0)$. The circumcenter O of Ω is the intersection of the perpendicular bisectors of $\triangle APC$, which has x-coordinate $\frac{p+14}{2}$. The y-coordinate is $\frac{3p+33}{8}$, so the common distance from O to each of A, P, and C is

$$\sqrt{\left(\frac{p-14}{2}\right)^2 + \left(\frac{3p+33}{8}\right)^2} = \frac{5}{8}\sqrt{p^2 - 10p + 169}.$$

With $Q = \left(\frac{15}{4}, 9\right)$, the distance OQ is $\sqrt{\left(\frac{2p+13}{4}\right)^2 + \left(\frac{3p-39}{8}\right)^2}$, so we get

$$\frac{25}{64}(p^2-10p+169) = \frac{4p^2+52p+169}{16} + \frac{9p^2-234p+1521}{64} \implies p = \frac{507}{56} \implies m+n = 507+56 = \boxed{563}.$$

18. A 3×3 grid of squares, filled in with 5 X's and 4 O's, is called *achievable* if it corresponds to some sequence of nine alternating X and O placements on the grid, beginning and ending with X's, such that at no point before the last square is filled in does the grid contain three X's or O's in a single row, column, or diagonal. For each achievable 3×3 tic-tac-toe grid, define its *connectedness* as the total number of squares in contiguous blocks of X's and O's that are not singletons. For example, the following tic-tac-toe grid is achievable and has connectedness 8:

X	X	O
X	O	O
X	O	X

Compute the expected connectedness of an achievable 3×3 grid with 5 X's and 4 O's chosen uniformly at random. Express your answer as a common fraction.

Answer: $\boxed{\frac{183}{26}}$

Throughout this solution, we will label the squares from 1-9, first from left to right, then from top to bottom, as follows:

1	2	3
4	5	6
7	8	9

First note (to facilitate the upcoming casework) that the only non-achievable grids are those where both X and O win: there are $12 \cdot 3 = 36$ of these, since there are $3 \cdot 2 = 6$ row placements for one row each of 3 X's and 3 O's, and likewise 6 column placements for two columns of X's and O's, as well as 3 ways to choose the rest of the 3 squares to be 2 X's and 1 O. We also have 12 achievable grids in which only O wins in a diagonal (2 choices for the diagonal, and 6 positions for the remaining O, none of the resulting grids which are overcounted), which is possible if and only if the grid is achievable (if the O's are in a row or column, no matter where the fourth O is placed, the five X's will form some complete row or column, hence making the grid unachievable; this logic is of course reversible). Of these, one can check that 16 grids are stalemates, and 62 should be actual wins for X; we now verify this explicitly.

If X wins via a complete row or column, we have 6 ways to choose this row or column. We then need exactly one X in each of the remaining two rows/columns (otherwise, O would have a complete row or column, which makes the game unachievable as above). This gives $6 \cdot 3^2 = 54$ choices, except 9 of these are double-counted: namely, if X wins *both* a row and column at the same time (in an L or T shape). These all give connectednesses of 9, except for the case where X has 13579 and O has 2468, in which case the connectedness is 5 – so we subtract $9 \cdot 8 + 5 = 77$ at the end of our computations.

Among each of the 45 resulting configurations, we can perform casework on which column (or row) X wins. If X wins from the first or third column, we have connectednesses of 9, 7, and 8 from the first X in spot 2 and letting the second X be in positions 3, 6, and 9; connectednesses of 7, 9, and 7 from an

X in position 5; and connectedness of 8, 7, and 9 from an X in position 8. These all sum to 71. On the other hand, if X wins in the second column, we can check that the sum of connectednesses is similarly $9+7+9+7+5+7+9+7+9=69$. By symmetry, we get a total of $2(71+69+71)-77=345$ connectedness if X wins decisively with a row or column.

If, on the other hand, X wins with a diagonal, we have 8 connectedness from any of the 8 cases where we either have a single contiguous block of X's or a square with a singleton X touching it diagonally, 5 connectedness from any of the 8 other cases, and 0 connectedness in the case 13579 for X, 2468 for O. This contributes a total of 104 additional connectedness.

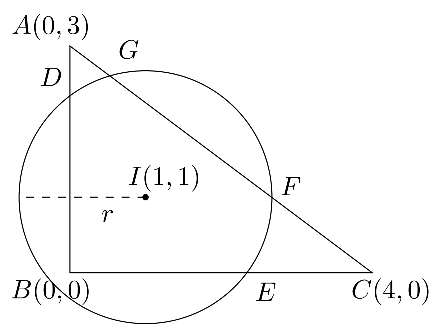
Finally, if we have one of the 16 possible stalemates, we can manually check the connectedness of each one: this does require a bit of brute force, but there exist shortcuts to find the connectedness rather quickly (which we leave to you to explore further), as well as symmetry properties similar to the ones we have exploited for the diagonal case. Altogether, the connectednesses will sum to 100 for stalemates (all are either 6 or 7).

In total, we have 549 connectedness from 78 achievable grids, so the expected connectedness of an achievable grid is $\boxed{\frac{183}{26}}$.

19. Triangle ABC has $AB=3$, $BC=4$, and $CA=5$. A circle O centered at the incenter I of $\triangle ABC$ has radius $r\in(\sqrt{2},\sqrt{5})$ so that the points D, E, F, and G of intersection of O with $\overline{AB}$, $\overline{BC}$, $\overline{CA}$, and $\overline{CA}$, respectively, are vertices of pentagon $DBEFG$ with area 5. Then $r^2=\frac{m-n\sqrt{p}}{q}$, where m, n, p, and q are positive integers with $\gcd(m,n,q)=1$ and p square-free. Compute $m+n+p+q$.

Answer: $\boxed{537}$

We draw a diagram as follows:



The equation of circle O is $(x-1)^2+(y-1)^2=r^2$, since (with the coordinate labels in the diagram above, without loss of generality) the incenter has coordinates $I=(1,1)$. Then $D=(0,\sqrt{r^2-1}+1)$ and $E=(\sqrt{r^2-1}+1,0)$. To determine the coordinates of F, we observe that, for points (x,y) lying on $\overline{AC}$, $y=3-\frac{3x}{4}$, so in particular, $(x-1)^2+\left(2-\frac{3}{4}x\right)^2=r^2$. Expanding, we get $\frac{25}{16}x^2-5x+5=r^2$, or $\frac{25}{16}x^2-5x+(5-r^2)=0$. Then by the quadratic formula, $x=\frac{8\pm4\sqrt{r^2-1}}{5}$, so $y=\frac{9\mp3\sqrt{r^2-1}}{5}$.

Since the area of $DBEFG$ is 5, and the area of $\triangle ABC$ is 6, the sum of the areas of $\triangle ADG$ and $\triangle FEC$ is 1, meaning that we have $AD\cdot\frac{8-4\sqrt{r^2-1}}{5}+EC\cdot\frac{9-3\sqrt{r^2-1}}{5}=2$. We know that $AD=3-(\sqrt{r^2-1}+1)=2-\sqrt{r^2-1}$ and that $EC=4-(\sqrt{r^2-1}+1)=3-\sqrt{r^2-1}$, so we end up with the equation $(3-\sqrt{r^2-1})(9-3\sqrt{r^2-1})+(2-\sqrt{r^2-1})(8-4\sqrt{r^2-1})=10$, or $3(3-\sqrt{r^2-1})^2+4(2-\sqrt{r^2-1})^2=10$.

Making the substitutions $a:=3-\sqrt{r^2-1}$ and $b:=2-\sqrt{r^2-1}$, so that $a=b+1$, we get the system of equations

$$a = b + 1, 3a^2 + 4b^2 = 10.$$

Solving directly, we obtain

$$3(b+1)^2 + 4b^2 = 10 \implies 7b^2 + 6b - 7 = 0 \implies b = \frac{-3 \pm \sqrt{58}}{7},$$

but since $r < \sqrt{5}$, we have $\sqrt{r^2 - 1} < 2$, and so $b > 0$; we discard the negative solution and take $b = \frac{-3+\sqrt{58}}{7}$. Henceforth, $2 - \sqrt{r^2-1} = \frac{-3+\sqrt{58}}{7}$, from which we get $r^2 = \frac{396-34\sqrt{58}}{49}$. Finally, $m + n + p + q = 396 + 34 + 58 + 49 = \boxed{537}$.

20. Giorgio takes a walk along a 4×4 square grid, beginning at the top-left square. At each square, he chooses to walk to each of the squares adjacent to him with equal probability, but he can only walk down and to the right. Once he reaches the lower-right corner of the grid, he stops. Giorgio's walk is then interpreted as a 4×4 matrix, with all squares that Giorgio has visited (including his starting point) being 1s and all squares that Giorgio has never visited being 0s.

    Define the *anti-Dyck-ness* $D(A)$ of the resulting matrix A as follows. Let B be the 3×3 matrix whose elements on or below the diagonal are exactly the same as those *strictly* below the diagonal of A, and $B = B^t$. (Here we denote by B^t the transpose of B; i.e. the matrix whose rows are the columns of B and whose columns are the rows of B, obtained by flipping B over its diagonal.) We say that $D(A)$ is equal to the determinant of B.

    For example, if Giorgio takes 2 steps down, 1 step right, 1 step down, and 2 steps right, we will have

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix},$$

    and thus,

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

    In this case, $D(A) = \det(B) = -1$.

    Compute the probability that $D(A) = 0$. Express your answer as a common fraction.

    Answer: $\boxed{\frac{9}{10}}$

    Among the $\frac{1}{4+1}\binom{8}{4} = 14$ "pseudo-Dyck" paths of length 6 (lying above the partial diagonal $\{a_{21}, a_{32}, a_{43}\}$), exactly one (DRDRDR) has $D(A) = 1$, and the other 13 have $D(A) = 0$. (It is left to the reader to check that this is actually the same as the number of Dyck paths for a matrix of dimension 1 larger.) This is because the entries of A below this partial diagonal are all zeros, and we get a diagonal matrix as a result (whose determinant is just the product of its diagonal entries). That is, if all diagonal elements are 1, the determinant is 1, and otherwise it is zero.

    We now consider the other $\binom{6}{3} - 14 = 6$ paths/matrices formed from these paths. This is easy enough to brute force: one can calculate 5 of the determinants to be 0 and the other to be -1 (using the fact that a row interchange flips the sign of the determinant, and one can add/subtract rows from each other with no change to the determinant, and by symmetry from the example given in the problem statement), giving a probability of $\boxed{\frac{9}{10}}$.