Online Open Tournament

Contest Solutions (Asia/Australia)

High School Division

Saturday, March 26, 2022

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1. Suppose that $(\log_2(x))^2 + \log_4(x) = 68$. Compute the product of the possible real values of x. Express your answer as a common fraction in simplest radical form.

   Answer: $\boxed{\frac{\sqrt{2}}{2}}$

   By log properties, $\log_4(x) = \frac{1}{2}\log_2(x)$. Substituting $a := \log_2(x)$, we obtain $a^2 + \frac{1}{2}a = 68$, or $2a^2 + a - 136 = 0$. Solving for a gives $a = \frac{-1\pm\sqrt{1089}}{4} = \frac{-1\pm 33}{4} = -8.5, 8$. We are looking for the product of the possible values of x, or, in other words, 2 to the power of the sum of the possible values of the base-2 logarithm of x by log laws. This is $2^{-0.5} = \boxed{\frac{\sqrt{2}}{2}}$ (and the exact values of x are $\frac{\sqrt{2}}{512}$ and 256, respectively).

2. Siya starts counting backward from 95 in increments of 4. As soon as she gets to a negative number, she starts counting upward in increments of 5. As soon as she reaches a number above 100, she counts backwards in increments of 6, and so forth. What is the 100^{th} number Siya counts?

   (a) 51
   (b) 54
   (c) 65
   (d) 69
   (e) none of the above

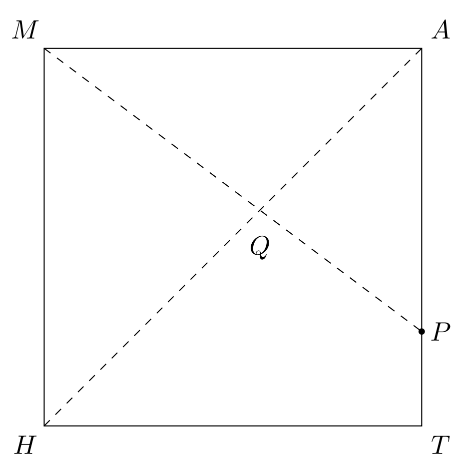
   Answer: $\boxed{D}$

   The 25^{th} number Siya will count is -1; the 46^{th} number is 104; the 64^{th} is -4; the 79^{th} is 101; the 92^{nd} is -3; and now we count upwards in increments of 9. The 100^{th} number Siya counts is $-3 + (100 - 92) \cdot 9 = \boxed{69}$.

3. Let $MATH$ be a unit square. Point P lies on $\overline{AT}$ with $MP = \frac{5}{4}$, and point Q is the intersection of the bisector of $m\angle MAT$ with $\overline{MP}$. Compute the area of triangle MAQ. Express your answer as a common fraction.

   Answer: $\boxed{\frac{3}{14}}$

   We have the following diagram:



Here $AP = \frac{3}{4}$ by the Pythagorean theorem. Since $\triangle AQP$ and $\triangle QMH$ are similar, $\frac{QA}{QH} = \frac{3}{4}$, and $\frac{HQ}{HA} = \frac{4}{7}$. Hence, the length of the altitude from Q to $\overline{MA}$ is $\frac{3}{7}$, and the area of $\triangle MAQ$ is $\boxed{\dfrac{3}{14}}$.

4. What is the value of

$$\sqrt{\frac{7+\sqrt{37}}{2}} + \sqrt{\frac{7-\sqrt{37}}{2}},$$

in simplest radical form?

Answer: $\boxed{\sqrt{7+2\sqrt{3}}}$

Call the desired expression S. We find that $S^2 = \left(\frac{7+\sqrt{37}}{2}\right) + 2\sqrt{\left(\frac{7+\sqrt{37}}{2}\right)\left(\frac{7-\sqrt{37}}{2}\right)} + \left(\frac{7-\sqrt{37}}{2}\right) = 7 + 2\sqrt{\frac{49-37}{4}} = 7 + 2\sqrt{3}$. This implies that $S = \boxed{\sqrt{7+2\sqrt{3}}}$.

Alternatively, one may denote $r := \sqrt{\frac{7+\sqrt{37}}{2}}$ and $s := \sqrt{\frac{7-\sqrt{37}}{2}}$, so that $r^2 + s^2 = 7$ and $rs = \sqrt{3}$. Then $r^2 + s^2 = (r+s)^2 - 2rs$ implies that $r + s = \boxed{\sqrt{7+2\sqrt{3}}}$. Yet another (similar) method is to instead denote $r := \frac{7+\sqrt{37}}{2}$ and $s := \frac{7-\sqrt{37}}{2}$, which we can readily identify as the roots of $x^2 - 7x + 3$ from $r + s = 7$ and $rs = 3$. We want $\sqrt{r} + \sqrt{s}$, which, when squared, yields $r + s + 2\sqrt{rs} = 7 + 2\sqrt{3}$; hence $\sqrt{r} + \sqrt{s} = \boxed{\sqrt{7+2\sqrt{3}}}$ as before.

5. Triangle ABC has $AB = 13$, $BC = 14$, and $CA = 15$. Point D is the foot of the perpendicular from A to $\overline{BC}$. Square $DEFG$ is drawn inside triangle ADC, with E on $\overline{AD}$, F on $\overline{AC}$, and G on $\overline{BC}$. Then $AF^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

Answer: $\boxed{6386}$

Let $A = (5, 12)$, $B = (0, 0)$, $C = (14, 0)$, without loss of generality. Then $F = (5 + s, s)$, and lies on the line $y = \frac{56-4x}{3}$, so $s = \frac{56-4(5+s)}{3}$ and $s = \frac{36}{7}$. It follows that $F = \left(\frac{71}{7}, \frac{36}{7}\right)$ and that $AF^2 = \left(\frac{71}{7}\right)^2 + \left(\frac{36}{7}\right)^2 = \frac{6337}{49}$, so that $m + n = \boxed{6386}$.

6. Jules wants to compute the value of $14_b \cdot 62_b$ in an integer base $b \geq 7$, but accidentally computes $41_b \cdot 26_b$ instead. For each positive integer $b \geq 7$, let $f(b)$ be the positive difference between the incorrect value and the correct value. Compute the remainder when $f(7) + f(8) + f(9) + f(10) + \cdots + f(100)$ is divided by 1000.

(a) 96
(b) 259
(c) 447
(d) 676
(e) none of the above

Answer: $\boxed{E}$

In general, the correct value will be $(b+4)(6b+2) = 6b^2 + 26b + 8$, while the incorrect value will be $(4b+1)(2b+6) = 8b^2 + 26b + 6$, so Jules' positive difference will be $f(b) = 2b^2 - 2$ for all $b \geq 7$. We then have

$$f(7) + f(8) + f(9) + f(10) + \cdots + f(100) = 2\left(\sum_{b=7}^{100} b^2\right) - 188$$

$$= 2\left(\sum_{b=1}^{100} b^2 - \sum_{b=1}^{6} b^2\right) - 188$$
$$= 2\left(\frac{100 \cdot 101 \cdot 201}{6} - \frac{6 \cdot 7 \cdot 13}{6}\right) - 188$$
$$= 2\left(50 \cdot 101 \cdot 67 - 7 \cdot 13\right) - 188$$
$$= 2 \cdot 338259 - 188 \equiv 2 \cdot 259 - 188 \equiv \boxed{330} \bmod 1000.$$

7. For how many ordered pairs (x, y) of positive integers with $x, y \leq 100$ is $xy + 2x + 3y$ a multiple of 6?

Answer: $\boxed{4200}$

By Simon's favorite factoring trick, we can rewrite $xy + 2x + 3y$ as $(x + 3)(y + 2) - 6$. So it suffices for $(x + 3)(y + 2)$ to be a multiple of 6, meaning it must be a multiple of both 2 and 3. Note that $(x + 3)(y + 2)$ is a multiple of 2 as long as either x is odd, y is even, or both. It is a multiple of 3 when either of the following are true: $x \equiv 0 \bmod 3$; or $x \equiv 1, 2 \bmod 3$ and $y \equiv 1 \bmod 3$. In the former case, assuming $x \equiv 0 \bmod 3$, the ordered pairs (x, y) such that x is odd are those with $x \in \{3, 9, 15, 21, \cdots, 99\}$ (which has 17 elements), of which there are $17 \cdot 100 = 1700$. Again assuming $x \equiv 0 \bmod 3$, if x is instead even, such that $x \in \{6, 12, 18, 24, \cdots, 96\}$ with 16 possible values, then y must be even, giving $16 \cdot 50 = 800$ additional ordered pairs.

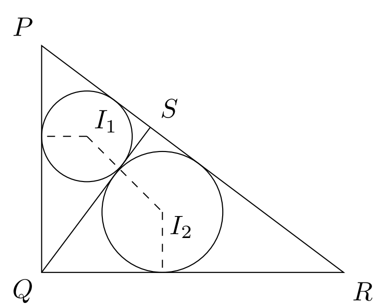
In the case that x is a non-multiple of 3, but $y \equiv 1 \bmod 3$, the number of such ordered pairs with x odd (giving $50 - 17 = 33$ possible values of x) is $33 \cdot 34 = 1122$. If x is an even non-multiple of 3, and $y \equiv 1 \bmod 3$ is even, then we have $50 - 16 = 34$ possible values of x, and $y \in \{4, 10, 16, 22, \cdots, 100\}$, giving rise to 17 possible values of y, hence an additional $34 \cdot 17 = 578$ ordered pairs (x, y).

In total, we have $1700 + 800 + 1122 + 578 = \boxed{4200}$ ordered pairs (x, y) with $1 \leq x, y \leq 100$ such that $(x + 3)(y + 2)$ is a multiple of 6.

8. Triangle PQR has $PQ = 3$, $QR = 4$, $RP = 5$, and a right angle at Q. Let S lie on $\overline{PR}$ so that $\overline{QS}$ is an altitude of $\triangle PQR$. Compute the smallest possible distance between two points lying on the inscribed circles of $\triangle PQS$ and $\triangle SQR$.

Answer: $\boxed{\sqrt{2} - \frac{7}{5}}$

It suffices to compute the inradii of each of $\triangle PQS$ and $\triangle SQR$, and then apply the Pythagorean theorem, as in the following diagram:



The altitude QS has length $\frac{3 \cdot 4}{5} = \frac{12}{5}$, so by the Pythagorean theorem, $PS = \frac{9}{5}$ and $SR = \frac{16}{5}$. By the inradius formula $A = rs$, where A denotes the area of the triangle and s denotes the semi-perimeter of the triangle, we get that the inradii of $\triangle PQS$ and $\triangle QRS$ are $\frac{3}{5}$ and $\frac{4}{5}$, respectively (also noting that they are similar to a 3-4-5 right triangle (whose inradius is 1), with respective scale factors $\frac{3}{5}$ and $\frac{4}{5}$).

Calling the incenters I_1 and I_2 respectively, and denoting by F_1 and F_2 the feet of their perpendiculars to $\overline{PS}$ and $\overline{SR}$ respectively, we notice that $m\angle SF_1I_1 = m\angle SF_2I_2 = 90°$, and so $SI_1 = \frac{3}{5}\sqrt{2}$ and $SI_2 = \frac{4}{5}\sqrt{2}$. We then get that $I_1I_2 = \sqrt{2}$ since $m\angle I_1SI_2 = 90°$. Subtracting the inradii gives the minimal distance of $\boxed{\sqrt{2} - \frac{7}{5}}$.

9. Perpendicular lines ℓ_1 and ℓ_2, whose y-intercepts sum to 22, intersect at the point $(5, -4)$. The largest possible value for one of the two y-intercepts can be written in the form $p + q\sqrt{r}$, where p, q, and r are positive integers and r is not divisible by the square of a prime. Compute $p + q + r$.

   Answer: $\boxed{47}$

   Let the y-intercepts of ℓ_1 and ℓ_2 be k and $22 - k$, respectively. Starting with ℓ_1, we can compute the slope to be $\frac{(-4)-k}{5-0} = \frac{-4-k}{5}$, and likewise, we can compute the slope of ℓ_2 to be $\frac{-26-k}{5}$. As ℓ_1 and ℓ_2 are perpendicular, we obtain $\frac{-4-k}{5} \cdot \frac{-26-k}{5} = -1$, so $(-4-k)(-26-k) = (4+k)(26+k) = -25$. This yields $k^2 + 30k + 129 = 0$, from which we get $k = -15 \pm 4\sqrt{6}$, and $22 - k$ will be the larger of the two y-intercepts (since $k < 11$). Namely, if we take $k = -15 - 4\sqrt{6}$, then we maximize $22 - k$ at $37 + 4\sqrt{6}$. Hence, $p + q + r = 37 + 4 + 6 = \boxed{47}$.

10. Leatrice writes down a 3-digit base-5 positive integer, chosen uniformly at random. Nora then chooses one of the digits in Leatrice's number uniformly at random and erases it. Compute the probability that the resulting number is a valid base-3 number.

    Answer: $\boxed{\frac{4}{15}}$

    Leatrice's original number may have contained at most one digit that is either 3 or 4; otherwise, even if one of the 3 or 4 digits were erased, there would still be at least one remaining, but 3 and 4 are not valid digits in base 3. The number of 3-digit base-5 positive integers with exactly one instance of either the digit 3 or 4 is $2 \cdot 3^2 + 2 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 2 = 42$, considering the cases where that digit is the hundreds, tens, or units digit (if it is the hundreds digit, then we have 2 choices for that hundreds digit, 3 choices for the tens digit, and 3 choices for the units digit; likewise, if it is the tens digit, we have 2 choices for the hundreds digit, 2 choices for the tens digit, and 3 choices for the units digit; finally, if it is the units digit, we again have 2 choices for the hundreds digit, 3 choices for the tens digit, and 2 choices for the units digit). Among these, 6 contain the 3 or 4 in the hundreds place and a 0 in the tens place, and 4 contain the 3 or 4 in the units place and a 0 in the tens place. There are also $2 \cdot 3^2 = 18$ numbers with no digits that are 3 or 4, of which 6 have a zero in the tens place.

    If Leatrice's number contains exactly one 3 or 4, the probability that erasing that digit will produce a valid base-3 number is $\frac{1}{3}$ if the tens digit is nonzero, and zero if the tens digit is zero (because the new number cannot begin with a zero). Otherwise, if there are no 3's or 4's in Leatrice's number, and the tens digit is not zero, it is guaranteed that the resulting base-3 number will be valid, no matter which digit is erased. If the tens digit is zero, the probability is $\frac{2}{3}$ (any digit except the hundreds digit can be erased). Altogether, since there are a total of $4 \cdot 5^2 = 100$ three-digit integers in base 5, we obtain a probability of $\frac{1}{3} \cdot \frac{32}{100} + 0 \cdot \frac{10}{100} + 1 \cdot \frac{12}{100} + \frac{2}{3} \cdot \frac{6}{100} = \boxed{\frac{4}{15}}$.

11. What is the units digit of

    $$\binom{2022}{1} + \binom{2021}{2} + \binom{2020}{3} + \binom{2019}{4} + \cdots + \binom{1013}{1010} + \binom{1012}{1011}?$$

    Answer: $\boxed{2}$

    We prove the following lemma:

Lemma 1.

$$\sum_{k\geq 0}\binom{n-k}{k}=F_{n+1}.$$

Proof. We prove this by induction; the base cases are left to the reader. For the inductive step, we want to show that

$$\sum_{k\geq 0}\binom{n+1-k}{k}=F_{n+2}.$$

By Pascal's identity,

$$\binom{n+1-k}{k}=\binom{n-k}{k}+\binom{n-k}{k-1},$$

and

$$\binom{n-k}{k-1}=\binom{(n-1)-(k-1)}{k-1}=\sum_{j\geq 0}\binom{n-1-j}{j}$$

by the hockey-stick identity. Therefore, by our inductive hypothesis, we have

$$\sum_{k\geq 0}\binom{n+1-k}{k}=\sum_{k\geq 0}\binom{n-k}{k}+\sum_{j\geq 0}\binom{n-1-j}{j}=F_{n+1}+F_n=F_{n+2}$$

as desired. □

It follows that the desired sum is $F_{2024}-1$, so it remains to compute the units digit of F_{2024}. By the Chinese remainder theorem, we want to compute F_{2024} mod 2 and F_{2024} mod 5. Since the Fibonacci terms follow the pattern OOEOOEOOE..., with every third term being even, and 2024 is not a multiple of 4, F_{2024} is odd, and is congruent to 1 mod 2. Modulo 5, the pattern is

$$1,1,2,3,0,3,3,1,4,0,4,4,3,2,0,2,2,4,1,0,\cdots,$$

with period 20. Thus, $F_{2024}\equiv F_4\equiv 3 \bmod 5$, and by CRT, $F_{2024}\equiv 3 \bmod 10$, so that $F_{2024}-1\equiv \boxed{2} \bmod 10$.

12. On moving day, Suri is packing five cubical boxes (each of which is long enough to a side to contain all smaller boxes) into the moving truck. A larger box can contain all smaller boxes, except the smallest box must be nested in some other, larger box, since it has fragile contents. In how many ways can she pack the boxes into the truck?

    Answer: $\boxed{96}$

    We approach this problem recursively. Let $S(n)$ be the number of ways for Suri to pack n boxes *without* the condition that the smallest box be nested in some other, larger box, for now. (For simplicity's sake, let's label the boxes 1 through n – these are not *literal* dimensions, but rather represent a hierarchy of sizes. The box with label 2, for example, might be 100 times as long to a side as box 1; what matters is that the box with label $n+1$ has side length no less than the sum of the side lengths of the boxes with label l for $1\leq l\leq n$.) Actually, we observe that the answer is in fact just $S(5)-S(4)$, since the packings with box 1 isolated are in bijection with the packings with four boxes altogether (boxes 2-5 are these four boxes, in particular).

    As our base cases, we obviously have $S(1)=1$ and $S(2)=2$ (either box 1 is inside box 2, or it is not). For $S(3)$, we can have both boxes 2 and 1 inside box 3, in which case we get $S(2)$ possible packings; box 2 may be inside box 3 but box 1 by its lonesome, which gives $S(1)S(1)=1$ possible packing (the first $S(1)$ term from the number of ways to pack 1 box (the box labeled 2) inside the largest "3" labeled box, and the second $S(1)$ term from the number of ways to pack the "1" box by itself. This may seem rather trivial for now, but this will become more interesting for the cases

$n = 4$ and $n = 5$); identically, box 1 may be inside box 3 but box 2 by its lonesome; or boxes 2 and 1 can both be outside box 3, in which case we have another $S(2) = 2$ potential packings. This gives $S(3) = 2 + 1 + 1 + 2 = 6$. Do we notice a pattern yet?

If not, let's try the next case. Similarly, to compute $S(4)$, we observe that the possible cases are to have all three smaller boxes inside box 4, for $S(3) = 6$ possible packings; exactly two out of the three smaller boxes inside box 4 and the other outside, which yields $\binom{3}{2} \cdot S(2) \cdot S(1) = 6$ packings; exactly one out of the three smaller boxes inside box 4 and the other two outside it, which yields $\binom{3}{1} \cdot S(1) \cdot S(2) = 6$ additional packings; and finally all three smaller boxes may lie outside $S(4)$, for another $S(3) = 6$ arrangements. This gives a total of $S(4) = 24$. *Now* do we notice a pattern? It *looks* as though $S(n) = n!$ in general; can we prove it? If we can, then we're done, because the answer is just $S(5) - S(4) = 5! - 4! = \boxed{96}$.

Following the method described for the cases $n = 3$ and $n = 4$, we can observe that

$$S(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} S(k)S(n-k-1).$$

That $S(n) = n!$ follows from a routine induction.

13. In triangle ABC, $AB = 6$, $BC = 8$, and $CA = 10$. Points P and Q lie on $\overline{AB}$ and $\overline{BC}$, respectively, with $PB = BQ = 2$, so that $\triangle APQ$ has circumcircle O which intersects $\overline{CA}$ at point R. Compute $\frac{AR}{AC}$. Express your answer as a common fraction.

Answer: $\boxed{\frac{22}{25}}$

We can let $A = (0, 6)$, $B = (0, 0)$, $C = (8, 0)$, $P = (0, 2)$, and $Q = (2, 0)$. The circumcenter of $\triangle APQ$ is at $(4, 4)$, so the equation of the circumcircle is

$$(x-4)^2 + (y-4)^2 = 20.$$

The intersection point with $\overline{CA}$, whose equation is $y = -\frac{3}{4}x + 6$, is the solution (x, y) to the system

$$y = -\frac{3}{4}x + 6, (x-4)^2 + (y-4)^2 = 20.$$

Substituting, we obtain

$$\begin{aligned} x^2 - 8x + 16 + \left(-\frac{3}{4}x + 6\right)^2 - 8\left(-\frac{3}{4}x + 6\right) + 16 = 20 \\ \implies \frac{25}{16}x^2 - 11x = 0 \\ \implies x = \frac{176}{25}, 0. \end{aligned}$$

But since $R \neq A$, we take $x = \frac{176}{25}$, and thus $\frac{AR}{AC} = \boxed{\frac{22}{25}}$.

14. N fair coins are flipped. If the expected product of the numbers of heads and tails flipped is an integer, compute the sum of all possible values of N less than or equal to 2022.

Answer: $\boxed{1022626}$

We aim to compute the expected product in terms of n, namely

$$E(n) := \sum_{k=0}^{n} \frac{\binom{n}{k}}{2^n} \cdot k(n-k).$$

Expanding the binomial coefficient, we have

$$E(n) = \frac{1}{2^n}\sum_{k=0}^{n}\frac{n!}{(k-1)!(n-k-1)!}$$

$$= \frac{1}{2^n}\sum_{k=0}^{n} n(n-1)\frac{(n-2)!}{(k-1)!(n-k-1)!}$$

$$= \frac{1}{2^n}\sum_{k=0}^{n} n(n-1)\binom{n-2}{k-1}$$

$$= \frac{1}{2^n}\sum_{k=0}^{n} n(n-1)2^{n-2} = \frac{n(n-1)}{4}.$$

For $E(n)$ to be an integer, we require that n be a multiple of 4 or one more than a multiple of 4. The sum of all such values less than or equal to 2022 iss $(1+4)+(5+8)+(9+12)+\cdots+(2017+2020)+2021 = (5+13+21+\cdots+4037)+2021 = 2021\cdot 505+2021 = 2021\cdot 506 = \boxed{1022626}$.

15. Triangle ABC in the xy-plane has $AB = 3$, $BC = 4$, and $CA = 5$, with $A = (0, 3)$, $B = (0, 0)$, and $C = (4, 0)$. Point I is the incenter of $\triangle ABC$. The smallest possible area of a rectangle with side lengths parallel to the coordinate axes circumscribing the circumcircles of $\triangle ABI$, $\triangle BCI$, and $\triangle CAI$ can be written in the form $\frac{m+n\sqrt{p}+q\sqrt{r}+s\sqrt{t}}{u}$, where all of the variables are positive integers, $\gcd(m, n, q, s) = 1$, and none of p, r, or t are divisible by the square of a prime number. Compute $m+n+p+q+r+s+t+u$.

Answer: $\boxed{283}$

Let $A = (0, 3)$, $B = (0, 0)$, and $C = (4, 0)$, so that $I = (1, 1)$ (from the inradius formula $[\triangle ABC] = rs$). Then we compute the equations of the circumcircles of each of the triangles $\triangle ABI$, $\triangle BCI$, and $\triangle CAI$. The circumcenter is the intersection of the perpendicular bisectors, so that of $\triangle ABI$ has coordinates $\left(-\frac{1}{2}, \frac{3}{2}\right)$, and hence the corresponding square of the circumradius is $\frac{5}{2}$; likewise for $\triangle BCI$ and $\triangle CAI$, whose circumcircles have equations $(x-2)^2+(y+1)^2 = 5$ and $\left(x-\frac{7}{2}\right)^2+\left(y-\frac{7}{2}\right)^2 = \frac{25}{2}$, respectively.

We now want to compute the product of the positive differences between the smallest and largest x- and y-coordinates of points lying on the three circumcircles. Setting $x = \frac{7}{2}$ to find the largest y-coordinate, we get $y = \frac{5\sqrt{2}+7}{2}$; the smallest y-coordinate, from $(x-2)^2+(y+1)^2 = 5$, is similarly $y = -\sqrt{5}-1$. The largest x-coordinate is also from $\left(x-\frac{7}{2}\right)^2+\left(y-\frac{7}{2}\right)^2 = \frac{25}{2}$, and setting $y = \frac{7}{2}$ again gives $x = \frac{5\sqrt{2}+7}{2}$; the smallest x-coordinate comes from $\left(x+\frac{1}{2}\right)^2+\left(y-\frac{3}{2}\right)^2 = \frac{5}{2}$, so that, from setting $y = \frac{3}{2}$, we get $x = \frac{-\sqrt{10}-1}{2}$.

Finally, the area of the circumscribing rectangle is

$$\left(\frac{5\sqrt{2}+7}{2}-(-\sqrt{5}-1)\right)\left(\frac{5\sqrt{2}+7}{2}-\frac{-\sqrt{10}-1}{2}\right) = \left(\frac{5\sqrt{2}+2\sqrt{5}+9}{2}\right)\left(\frac{5\sqrt{2}+\sqrt{10}+8}{2}\right)$$

$$= \frac{50+5\sqrt{20}+40\sqrt{2}+10\sqrt{10}+10\sqrt{2}+16\sqrt{5}+45\sqrt{2}+9\sqrt{10}+72}{4} = \frac{122+19\sqrt{10}+26\sqrt{5}+95\sqrt{2}}{4},$$

and it follows that $m+n+p+q+r+s+t+u = 122+19+10+26+5+95+2+4 = \boxed{283}$.

16. A fair six-sided die, labeled with the integers from 1 to 6 inclusive on its faces, is rolled until a 6 comes up for the first time. After the first roll of a 6, let $E(N)$ be the expected ratio of the product of all rolls up to that point to the quantity $N^{\text{total number of rolls }-1}$. If $E(N) = 2022$, then $N = \frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m+n$.

Answer: $\boxed{7076}$

Where k is the total number of rolls, we observe that

$$E(N) = \sum_{k=1}^{\infty} \left(\left(\frac{5}{6} \right)^{k-1} \cdot \frac{1}{6} \right) \cdot \frac{6 \cdot 3^{k-1}}{N^{k-1}},$$

since the expected value of each of the $k-1$ non-6 rolls preceding the 6 is 3. This simplifies to

$$E(N) = \sum_{k=1}^{\infty} \left(\frac{5}{2N} \right)^{k-1} = \frac{1}{1 - \frac{5}{2N}} = \frac{2N}{2N-5},$$

and $\frac{2N}{2N-5} = 2022 \implies N = \frac{5055}{2021}$, so $m + n = 5055 + 2021 = \boxed{7076}$.

17. Let $f(x)$ be a monic cubic polynomial with real coefficients whose roots sum to 9. Suppose that the reciprocals of the roots of $g(x) := f(x)^2$ sum to $-\frac{1}{3}$. Compute the coefficient of the linear term of $f(x)$ for which the value of $f(1)+g(1)$ attains its minimum. Express your answer as a common fraction.

Answer: $\boxed{\frac{15}{14}}$

Letting $f(x) := x^3 - 9x^2 + ax + b$ in general (where the coefficient of x^2 is -9 by Vieta's formulas), and computing

$$g(x) = x^6 - 18x^5 + (81 + 2a)x^4 + x^3(2b - 18a) + x^2(a^2 - 18b) + x(2ab) + b^2,$$

we find that $-\frac{2ab}{b^2} = -\frac{2a}{b} = -\frac{1}{3} \implies b = 6a$. Thus, $f(x) = x^3 - 9x^2 + ax + 6a$, $f(1) = -8 + 7a$, and $f(1) + g(1) = (-8 + 7a)(-7 + 7a)$, which attains its minimum where $-7 + 7a = \frac{1}{2} \implies a = \boxed{\frac{15}{14}}$.

18. Suppose that $(a_i)_{i \geq 1}$ is a sequence such that $a_1 = 2$, $a_2 = 6$, $a_3 = 24$, and for all $n \geq 4$, $a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}$. Compute the sum of all positive integers $n \leq 1000$ for which $a_n - 8$ is a power of 2.

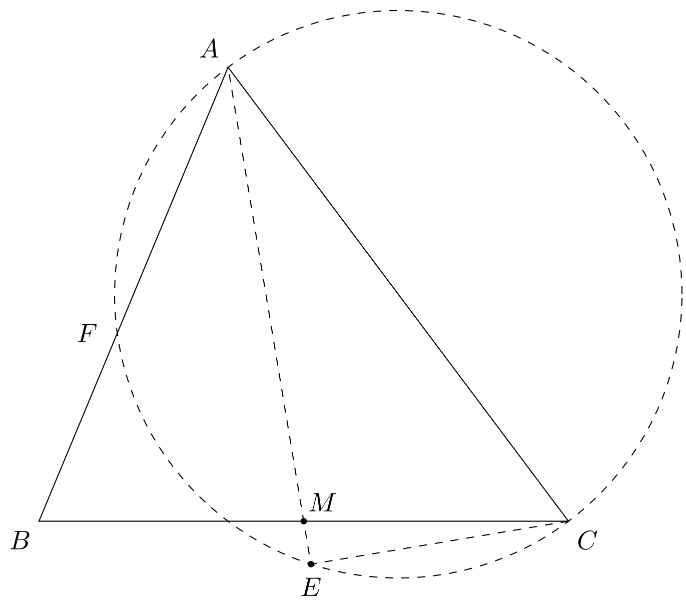
Answer: $\boxed{225}$

The characteristic equation of a_n (when the recursion is written as $a_n - 5a_{n-1} + 8a_{n-2} - 4a_{n-3} = 0$) is $x^3 - 5x^2 + 8x - 4 = 0$, whose roots are 1, 2, and 2; hence, the closed form for a_n will be $A \cdot 1^n + B \cdot 2^n + Cn \cdot 2^n$ for some constants A, B, and C. From the initial conditions $a_1 = 2$, $a_2 = 6$, and $a_3 = 24$, we get that $A + 2B + 2C = 2$, $A + 4B + 8C = 6$, and $A + 8B + 24C = 24$, respectively. Solving the system yields $(A, B, C) = \left(8, -\frac{11}{2}, \frac{5}{2}\right)$, so $a_n = 8 + (5n - 11)2^{n-1}$. For this to be 8 more than a power of 2, $5n - 11$ must also be a power of 2, meaning that such a power of 2 is congruent to 4 mod 5. The congruence $2^k \equiv 4 \bmod 5$ has solutions $k \equiv 2 \bmod 4$, so $5n - 11 = 4, 64, 1024$ (but $5n - 11 = 16384 \implies n > 1000$. So $n = 3, 15, 207$ are the corresponding values of n under 1000, and these sum to $\boxed{225}$.

19. Let ABC be a triangle with side lengths $AB = 13$, $BC = 14$, and $CA = 15$. Let M be the midpoint of $\overline{BC}$, and let E be the projection of C onto the extension of $\overline{AM}$. The circumcircle of $\triangle ACE$ intersects $\overline{AB}$ at point $F \neq A$. Compute $\frac{BF}{BA}$. Express your answer as a common fraction.

Answer: $\boxed{\frac{70}{169}}$

Without loss of generality, let $A = (5, 12)$, $B = (0, 0)$, and $C = (14, 0)$:



As $m\angle AEC = 90^\circ$, the circumcenter of $\triangle ACE$ lies at the midpoint of $\overline{AC}$, which is $\left(\frac{19}{2}, 6\right)$. Let $F = (5x, 12x)$ for some $0 < x < 1$; it follows that $\left(5x - \frac{19}{2}\right)^2 + (12x - 6)^2 = \left(\frac{15}{2}\right)^2$, or $169x^2 - 239x + 70 = 0$. Solving for x, we obtain $x = \frac{239 \pm \sqrt{239^2 - 169 \cdot 280}}{338} = \frac{239 \pm 99}{338} = \frac{70}{169}, 1$. Since $F \neq A$, we have $x = \frac{BF}{BA} = \boxed{\frac{70}{169}}$.

20. Suppose that z_1, z_2, and z_3 are complex numbers that are the roots of some cubic polynomial with positive integer coefficients and leading coefficient 1 such that

$$\frac{z_1 + 1}{z_1^2} + \frac{z_2 + 1}{z_2^2} + \frac{z_3 + 1}{z_3^2} = -\frac{21}{16}$$

and

$$z_1^2 + z_2^2 + z_3^2 = 73.$$

Compute the smallest possible value of $|z_1^3 + z_2^3 + z_3^3|$.

Answer: $\boxed{669}$

Let z_1, z_2, and z_3 be the roots of $x^3 + kx^2 + lx + m$; then $z_1^2 + z_2^2 + z_3^2 = (z_1 + z_2 + z_3)^2 - 2(z_1 z_2 + z_1 z_3 + z_2 z_3) = k^2 - 2l = 73$. Furthermore, we can write $\frac{z_1+1}{z_1^2} + \frac{z_2+1}{z_2^2} + \frac{z_3+1}{z_3^2}$ as $\left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right) + \left(\frac{1}{z_1}^2 + \frac{1}{z_2}^2 + \frac{1}{z_3}^2\right)$. Recall that the reciprocals of roots of a polynomial are precisely the roots of the polynomial with coefficients in the reverse order, so $\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z^3}$ is the sum of the roots of $mx^3 + lx^2 + kx + 1$, or $-\frac{l}{m}$.

Note also that

$$\left(\frac{1}{z_1}\right)^2 + \left(\frac{1}{z_2}\right)^2 + \left(\frac{1}{z_3}\right)^2 = \left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right)^2 - 2\left(\frac{1}{z_1 z_2} + \frac{1}{z_1 z_3} + \frac{1}{z_2 z_3}\right) = \left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right)^2 - 2\left(\frac{z_1 + z_2 + z_3}{z_1 z_2 z_3}\right)$$

$$= \left(-\frac{l}{m}\right)^2 - 2 \cdot \frac{-k}{-m} = \frac{l^2}{m^2} - \frac{2k}{m}.$$

Altogether, we get $-\frac{l}{m}+\frac{l^2}{m^2}-\frac{2k}{m}=-\frac{21}{16}$, or $(l+2k)m-l^2=\frac{21}{16}m^2$. This can be rearranged as $\frac{21}{16}m^2-(l+2k)m+l^2=0$, so that

$$m=\frac{(l+2k)\pm\sqrt{(l+2k)^2-\frac{21}{4}l^2}}{\frac{21}{8}}.$$

The important part here is that $(l+2k)^2=l^2+4lk+4k^2\geq\frac{21}{4}l^2$; that is, $4kl+4k^2\geq\frac{17}{4}l^2$, or

$$\frac{17}{4}l^2-4kl-4l^2\leq 0\implies 17l^2-16kl-16k^2\leq 0.$$

Since $l=\frac{k^2-73}{2}$ (from $k^2-2l=73$), we have

$$\frac{17}{4}(k^2-73)^2-8k(k^2-73)-16k^2\leq 0\implies 17k^4-32k^3-2546k^2+2336k+73^2\cdot 17\leq 0.$$

Some approximation shows that the only integers satisfying this inequality are $k=8$ and $k=9$, but we discard $k=8$ in favor of $k=9$, since $k^2-2l=73$ and k,l,m are positive integer coefficients.

Thus, $l=4$ and $m=16$, so $z_1^3+z_2^3+z_3^3=(z_1+z_2+z_3)^3-(3z_1^2(z_2+z_3)+3z_2^2(z_1+z_3)+3z_3^2(z_1+z_2)+6z_1z_2z_3)=(-9)^3-3z_1^2(-9-z_1)-3z_2^2(-9-z_2)-3z_3^2(-9-z_3)-6(-16)=-633+(3z_1^3+3z_2^3+3z_3^3)+27(z_1^2+z_2^2+z_3^2)$, and $z_1^3+z_2^3+z_3^3=-669$, with magnitude $\boxed{669}$ (and this is in fact the *only* possible value).