Online Open Tournament

Contest Solutions (North/Central/South America)

High School Division

Saturday, March 26, 2022

The problems and solutions for this competition were prepared by the teaching staff of CyberMath Academy.
Thanks to Thinula de Silva and Freya Edholm for reviewing these problems and solutions!

1. What is the sum of all positive integers that evenly divide 1680 but not 240?

   Answer: $\boxed{5208}$

   For a positive integer to divide $1680 = 240 \cdot 7$ but not 240, it must be seven times a factor of 240 (as 7 does not divide 240). The sum of all such factors is 7 times the sum of factors of $240 = 5 \cdot 3 \cdot 2^4$, which by the sum of factors formula is $(5^1 + 5^0)(3^1 + 3^0)(2^4 + 2^3 + 2^2 + 2^1 + 2^0) = 6 \cdot 4 \cdot 31 = 744$. The requested sum is therefore $7 \cdot 744 = \boxed{5208}$.

2. The product of 9 consecutive positive integers, the median of which is 7, is equal to t times the product of 5 consecutive positive integers, the median of which is 7. Compute t.

   (a) 280
   (b) 720
   (c) 1320
   (d) 2520
   (e) none of the above

   Answer: $\boxed{C}$

   The 9 consecutive integers must be 3 through 11, inclusive, and the 5 consecutive integers must be 5 through 9, inclusive. Thus, $t = 3 \cdot 4 \cdot 10 \cdot 11 = \boxed{1320}$.

3. Let ABC be a triangle with $AB = 20$, $BC = 22$, and $CA^2 = 444$. Compute the area of $\triangle ABC$. Express your answer in simplest radical form.

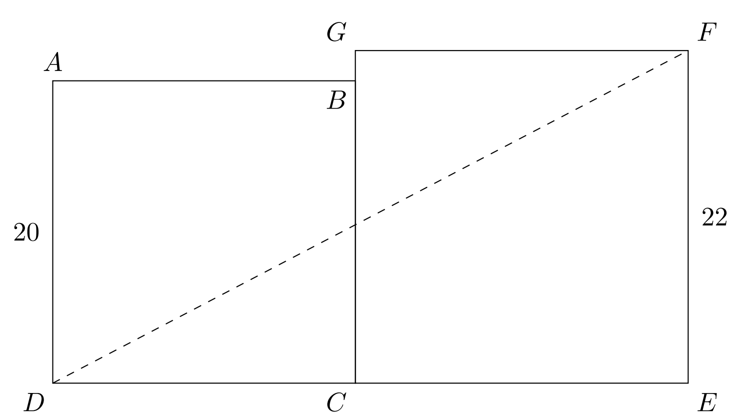
   Answer: $\boxed{110\sqrt{3}}$

   By the law of cosines, $CA^2 = AB^2 + BC^2 - 2 \cdot AB \cdot BC \cdot \cos(m\angle ABC)$, so $444 = 400 + 484 - 880\cos(m\angle ABC)$, and $\cos(m\angle ABC) = \frac{1}{2}$. Thus, $\sin(m\angle ABC) = \frac{\sqrt{3}}{2}$, and the area formula $\frac{1}{2}ab\sin C$ for side lengths a and b adjacent to a vertex angle measure C yields an area of $\frac{1}{2} \cdot 20 \cdot 22 \cdot \frac{\sqrt{3}}{2} = \boxed{110\sqrt{3}}$.

4. Square $ABCD$ has side length 20, and square $CEFG$ has side length 22, with $\overline{BC}$ lying in the interior of $\overline{CG}$. Compute the area of pentagon $ABGFD$. (Note: this problem was thrown out (and everyone received the point) due to multiple valid configurations by rotation. Below is the *originally intended* answer/solution for reference.)

   Answer: $\boxed{422}$

   We draw the diagram as follows:



The area below the dashed line segment $\overline{AF}$ is the area of the right triangle with leg lengths $DE = 20 + 22 = 42$ and $EF = 22$, or $42 \cdot 11 = 462$. Subtracting this from the area of the entire figure, which is $20^2 + 22^2 = 884$, gives the area above $\overline{DF}$, which is $\boxed{422}$.

5. A problem on a test has an answer that is a common fraction, and contains the instruction, "Let your answer be of the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$." If the raw answer were some positive real quantity r, the final answer would be some integer N, but if the raw answer were instead $r + \frac{1}{5}$, the final answer would be 5. Compute the sum of all possible values of N.

   (a) 75
   (b) 90
   (c) 110
   (d) 135
   (e) none of the above

   Answer: $\boxed{B}$

   The raw value of $r + \frac{1}{5}$ must be either $\frac{1}{4}$, $\frac{2}{3}$, $\frac{3}{2}$, or $\frac{4}{1}$, which yield $r = \frac{1}{20}$, $\frac{7}{15}$, $\frac{13}{10}$, and $\frac{19}{5}$, respectively, and thus, $N = 21, 22, 23, 24$, which sum to $\boxed{90}$.

6. Consider the set $S = \{1, 2, 3, \cdots, 1100\}$. What is the fewest number of elements that we must remove from S so that there is no pair of distinct elements in S that sum to 2022?

   Answer: $\boxed{89}$

   If two elements of the set sum to 2022, then at least one of them is 1012 or larger. By removing the 89 elements from 1012 through 1100, inclusive, we guarantee that no pair chosen from the new set can sum to 2022. Otherwise, if we remove 88 of 1012 through 1100 and leave some $1012 \leq x \leq 1100$, we can pair x with $922 \leq 2022 - x \leq 1010$, which remains in the new set. It follows that we must remove at least $\boxed{89}$ elements from the set S.

7. Compute the number of positive integers $n \leq 10000$ for which there lies a perfect square between n and $n + 100$, inclusive.

   Answer: $\boxed{7550}$

   Note that, for all positive integers $k \leq 49$, the positive difference between k^2 and $(k + 1)^2$ is less than 100. The positive differences for $50 \leq k \leq 99$ are the odd integers from 101 through 199, so there will be $0, 2, 4, \cdots, 98$ "gaps" of width 100 (101 including both endpoints), which sum to 2450 integers n for which there is no perfect square in $[n, n + 100]$. Thus, we have $10000 - 2450 = \boxed{7550}$ integers $n \leq 10000$ for which there is a perfect square in $[n, n + 100]$.

8. Given that $0 \leq x < 2\pi$ is a real number for which

$$\frac{\sin(3x)}{\sin(x)} = \frac{3}{4},$$

   the value of $\cos^2(x)$ can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

   Answer: $\boxed{23}$

   Recalling the identity $\sin(3x) = 3\sin(x) - 4\sin^3(x)$ (which can also be derived from the double-angle and/or sum formulas), we obtain $3 - 4\sin^2(x) = \frac{3}{4}$. Thus, $\sin^2(x) = \frac{9}{16}$, or $\cos^2(x) = \frac{7}{16}$. Then $m + n = 7 + 16 = \boxed{23}$.

9. Compute

$$\sum_{k=0}^{10} k\binom{10}{k}^2.$$

Answer: $\boxed{923780}$

Substituting $10-k$ in place of k in the expression for S, we get

$$\sum_{k=0}^{10}(10-k)\binom{10}{10-k}^2 = \sum_{k=0}^{10}(10-k)\binom{10}{k}^2,$$

so that

$$\sum_{k=0}^{10} k\binom{10}{10-k}^2 + (10-k)\binom{10}{10-k}^2 = 10\sum_{k=0}^{10}\binom{10}{k}^2 = S+S = 2S$$

$$\implies S = 5\sum_{k=0}^{10}\binom{10}{k}^2 = 5\binom{20}{10} = \boxed{923780}.$$

10. Regular hexagon $ABCDEF$ has side length 1. How many of the $\binom{6}{4} = 15$ quadrilaterals whose vertices are four distinct vertices of the hexagon have an inscribed circle?

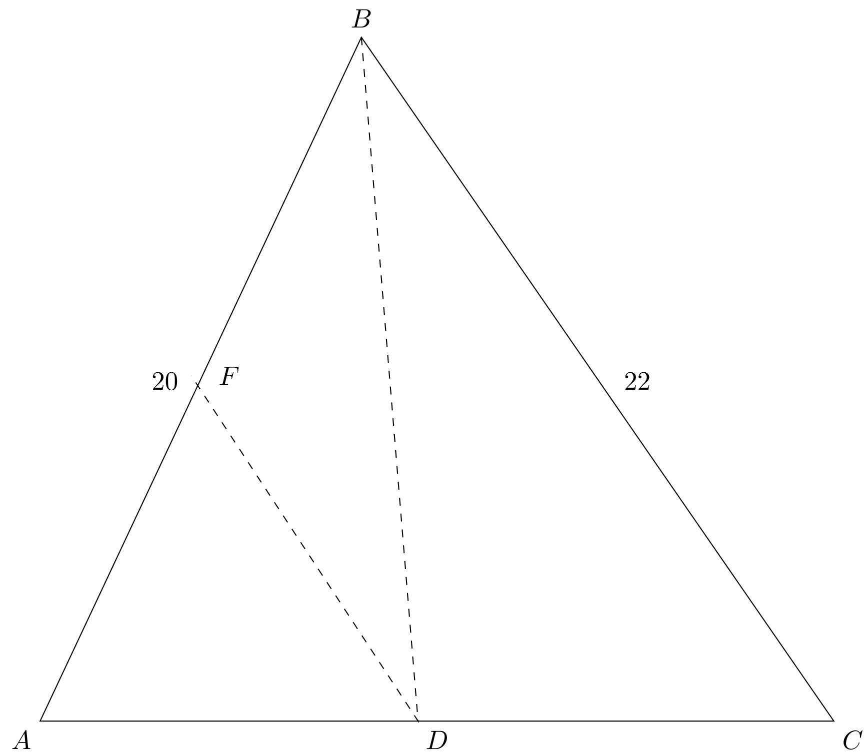
Answer: $\boxed{6}$

By Pitot's theorem, a quadrilateral $ABCD$ has an incircle if and only if $AB + CD = BC + DA$; i.e. if it is *tangential.* We have $ABCE$, $ABDF$, $ACDE$, $ACEF$, $BCDF$, and $BDEF$ (non-square rectangles, in particular, are not tangential), for a total of $\boxed{6}$ quadrilaterals (using symmetry and the fact that the height of the hexagon is the same as, say, the length of AC and its rotations/reflections). One can also apply a symetry argument upon considering the two vertices of the hexagon that are *not* vertices of the quadrilateral and arrive at the same result.

11. Triangle ABC has $AB = 20$, $BC = 22$, and $CA = 21$. Point D lies on $\overline{AC}$ such that $m\angle ABD = m\angle CBD$. Point F lies on $\overline{AB}$ such that $\overline{FD} \parallel \overline{BC}$. The area of $\triangle BDF$ can be written in the form $\frac{p\sqrt{q}}{r}$, where p, q, and r are positive integers such that $\gcd(p, r) = 1$ and q is square-free. Compute $p + q + r$.

Answer: $\boxed{1408}$

We draw the following diagram:



Because $\overline{FD}$ and $\overline{BC}$ are parallel, we have a homothety $\triangle AFD \sim \triangle ABC$ about point A with scale factor $\frac{10}{21}$, which is the ratio $\frac{AD}{AC}$ by the angle bisector theorem. The area of $\triangle BDF$ is thus equal to $\frac{10}{21}$ the area of $\triangle ABC$ (which is $\frac{21\sqrt{1311}}{4}$, by Heron's formula), minus the area of $\triangle AFD$, which is $\left(\frac{10}{21}\right)^2 = \frac{100}{441}$ times the area of $\triangle ABC$. In other words, it is $\frac{110}{441} \cdot \frac{21\sqrt{1311}}{4} = \frac{55\sqrt{1311}}{42}$, so that $p + q + r = 55 + 1311 + 42 = \boxed{1408}$.

12. Triangle ABC has $AB = 5$, $BC = 12$. For some real numbers $x \in [0, 1]$ and t, if $\cos(m\angle ABC) = x$, then $CA^2 = t$, but if $\sin(m\angle ABC) = x$, then $CA^2 = t - 24$. Compute t.

    Answer: $\boxed{97}$

    By law of cosines, $CA = \sqrt{169 - 120x}$ with $\cos(m\angle ABC) = x$, but if $\sin(m\angle ABC) = x$, i.e. $\cos(m\angle ABC) = \sqrt{1 - x^2}$, then $CA = \sqrt{169 - 120\sqrt{1 - x^2}}$, and $169 - 120x = (169 - 120\sqrt{1 - x^2}) + 24$, or $24 - 120\sqrt{1 - x^2} = -120x \implies 5\sqrt{1 - x^2} = 1 + 5x$. Squaring both sides, we get $25 - 25x^2 = 1 - 10x + 25x^2 \implies 50x^2 - 10x - 24 = 0 \implies x = -\frac{3}{5}, \frac{4}{5}$. We discard $x = -\frac{3}{5}$ (since $x \in [0, 1]$, as per the problem statement) in favor of $x = \frac{4}{5}$, and find that, if $\cos(m\angle ABC) = \frac{4}{5}$, then $CA^2 = 169 - 120 \cdot \frac{4}{5} = 73$, and if $\sin(m\angle ABC) = \frac{4}{5}$, then $CA^2 = 169 - 120 \cdot \frac{3}{5} = 97$. Indeed, $t = \boxed{97}$ then satisfies the problem statement (with $x = \frac{3}{5}$).

    Alternatively, one may apply the law of cosines directly to $\cos(m\angle ABC) = \frac{\sqrt{25-h^2}}{5}$, and get that $h^2 + (12 - \sqrt{25 - h^2})^2 = 169 + 24\sqrt{25 - h^2} \implies h = 4, -3$, so $h = 4$, then proceed as above.

13. Define the *A-index* of a permutation of a string of letters to be the total number of A's in contiguous substrings of the string that have length at least 2. For example, the A-index of the permutation AAALBMA of ALABAMA is 3, and the A-index of the permutation ABRCAADBARA of ABRACADABRA is 2. Compute the sum of all A-indices of all permutations of the string AAAABBCC.

    Answer: $\boxed{1080}$

If all four A's are consecutive, or if they come in two blocks of two, the A-index will be 4. The former case is attained by $5 \cdot \binom{4}{2} = 30$ permutations, while the latter is attained by 10 positions for the AA blocks multiplied with $\binom{4}{2} = 6$ ways to arrange the 2 B's and 2 C's in the remaining space, for 60 additional permutations, or 90 permutations in total with an A-index of 4. (Another method for this latter case is to observe that such a permutation must be of the form (j B's and C's)(A block)(k B's and C's)(A block)($4-k-j$ B's and C's); summing over $k = 1$ to $k = 4$, and then from $j = 0$ to $4-k$, we get $4+3+2+1 = 10$ positions for the A blocks.)

For an A-index of 3, there must be exactly three consecutive A's; this is achieved by $6(4+3+3+3+3+4) = 120$ permutations. For an A-index of 2, there must one block of two consecutive A's and two separate A's not adjacent to that block; we can count 30 ways to place the AA block, for a total of $6 \cdot 30 = 180$ permutations. The remaining permutations will have all A's non-adjacent, and hence A-index 0 (and indeed, we can check that there are $\frac{8!}{4!2!2!} - 90 - 120 - 180 = 30$ of these, for there are 5 ways to place the A's non-consecutively, and 6 ways to arrange the B's and C's). The sum of all A-indices is therefore $90 \cdot 4 + 120 \cdot 3 + 180 \cdot 2 = \boxed{1080}$.

14. Triangle ABC has $AB = 3$, $BC = 4$, and $CA = 5$. Point P lies on $\overline{BC}$ so that $\tan(m\angle BAP) + \tan(m\angle PAC) = 1$. Compute AP^2. Express your answer as a common fraction.

Answer: $\boxed{\dfrac{45}{4}}$

We have $m\angle BAP + m\angle PAC = m\angle BAC$, and $\tan(m\angle BAC) = \frac{4}{3}$. Recall that $\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$, so $\tan(m\angle BAP + m\angle PAC) = \frac{4}{3}$ implies $\tan(m\angle BAP)\tan(m\angle PAC) = \frac{1}{4}$.

If we let $x := \tan(m\angle BAP)$ and $y := \tan(m\angle PAC)$, solving the system of equations $x + y = 1$, $xy = \frac{1}{4}$ yields $(x, y) = \left(\frac{1}{2}, \frac{1}{2}\right)$, so in particular, $\tan(m\angle BAP) = \frac{1}{2}$. This means that $\frac{BP}{BA} = \frac{1}{2}$, or $BP = \frac{3}{2}$. Thus, $AP^2 = AB^2 + BP^2 = 3^2 + \left(\frac{3}{2}\right)^2 = \boxed{\dfrac{45}{4}}$.

15. Suppose that z_1 and z_2 are complex numbers with $|z_1| = |z_2| = 1$ and $z_1 + z_2 = 1 + \frac{3}{2}i$. Compute the product of the imaginary parts of z_1 and z_2. Express your answer as a common fraction.

Answer: $\boxed{\dfrac{105}{208}}$

Solution: Let $z_1 = a + bi$ and $z_2 = c + di$. From $|z_1| = |z_2| = 1$ we get that $a^2 + b^2 = c^2 + d^2 = 1$. Furthermore, $a + c = 1$ and $b + d = \frac{3}{2}$. We want to find the value of bd. Squaring both of the equations $a + c = 1$ and $b + d = \frac{3}{2}$, we find that $(a^2 + 2ac + c^2) + (b^2 + 2bd + d^2) = \frac{13}{4}$; rearranging, we get $(a^2+b^2+c^2+d^2)+2(ac+bd) = \frac{13}{4}$. Since $a^2+b^2+c^2+d^2 = 2$, it follows that $ac+bd = \frac{5}{8}$. As $c = 1-a$ and $d = 1-b$, this can be rewritten as $a(1-a) + b\left(\frac{3}{2} - b\right) = \frac{5}{8}$. Expanding, we get $a - a^2 + \frac{3}{2}b - b^2 = \frac{5}{8}$, so that $a + \frac{3}{2}b = \frac{13}{8}$, or, for the sake of convenience, $8a + 12b = 13 \implies 8a = 13 - 12b$. Then $64a^2 = 169 - 312b + 144b^2 = 64 - 64b^2$, so $208b^2 - 312b + 105 = 0$, which implies that $b = \frac{39 \pm 2\sqrt{39}}{52}$ (the sign choice is irrelevant) and that $d = \frac{39 \mp 2\sqrt{39}}{52}$. Finally, regardless of which signs are chosen (so long as they are opposite), we get that $bd = \boxed{\dfrac{105}{208}}$.

16. Evaluate the sum

$$\sum_{n=1}^{\infty} \frac{n^4}{n!}.$$

Answer: $\boxed{15e}$

Rewrite the sum S as

$$\frac{45}{2}+\sum_{n=4}^{\infty}\frac{n^3}{(n-1)!},$$

then re-index to obtain

$$S-\frac{45}{2}=\sum_{n=3}^{\infty}\frac{(n+1)^3}{n!}=\sum_{n=3}^{\infty}\frac{n^3}{n!}+\frac{3n^2}{n!}+\frac{3n}{n!}+\frac{1}{n!}.$$

We can evaluate each of these sums separately; the first sum can again be re-indexed in a recursive manner as

$$\begin{aligned}
\sum_{n=3}^{\infty}\frac{n^3}{n!}&=\sum_{n=3}^{\infty}\frac{n^2}{(n-1)!}\\
&=\sum_{n=2}^{\infty}\frac{(n+1)^2}{n!}\\
&=\sum_{n=2}^{\infty}\frac{n^2}{n!}+\frac{2n}{n!}+\frac{1}{n!}\\
&=\sum_{n=1}^{\infty}\frac{n+1}{n!}+\sum_{n=1}^{\infty}\frac{2}{n!}+(e-2)\\
&=\sum_{n=1}^{\infty}\frac{n}{n!}+\frac{1}{n!}+2(e-1)+(e-2)\\
&=\sum_{n=0}^{\infty}\frac{1}{n!}+(e-1)+2(e-1)+(e-2)=e+(4e-5)=5e-5;
\end{aligned}$$

$$\begin{aligned}
\sum_{n=3}^{\infty}\frac{3n^2}{n!}&=3\sum_{n=3}^{\infty}\frac{n}{(n-1)!}\\
&=3\sum_{n=2}^{\infty}\frac{n+1}{n!}\\
&=3\sum_{n=2}^{\infty}\frac{1}{(n-1)!}+\frac{1}{n!}\\
&=3\left(\sum_{n=1}^{\infty}\frac{1}{n!}+(e-2)\right)=3((e-1)+(e-2))=6e-9;
\end{aligned}$$

$$\begin{aligned}
\sum_{n=3}^{\infty}\frac{3n}{n!}&=3\sum_{n=3}^{\infty}\frac{1}{(n-1)!}\\
&=3\sum_{n=2}^{\infty}\frac{1}{n!}=3(e-2)=3e-6;
\end{aligned}$$

and, finally,

$$\sum_{n=3}^{\infty}\frac{1}{n!}=e-\frac{5}{2},$$

so in total,

$$S-\frac{45}{2}=15e-\frac{45}{2}\implies S=\boxed{15e}.$$

17. For each real number x, define

$$S(x) := ix - \frac{1}{2!}x^2 - i\frac{1}{3!}x^3 + \frac{1}{4!}x^4 + i\frac{1}{5!}x^5 - \frac{1}{6!}x^6 - i\frac{1}{7!}x^7 + \frac{1}{8!}x^8 + \cdots$$

If $0 \le x < 2\pi$ is a real number such that $S(x)^{12} = 1$, compute the product of the possible values of x.

Answer: $\boxed{\frac{5\pi^2}{9}}$

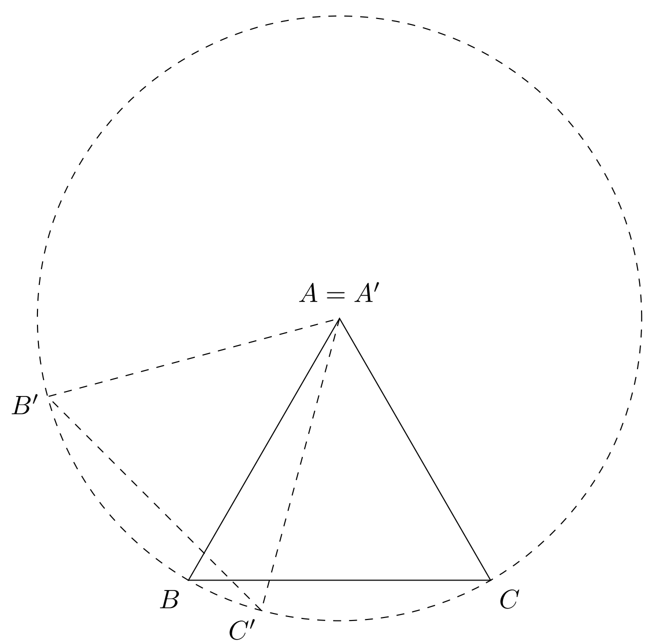
Observe that $S(x)+1$ is the Taylor expansion of e^x evaluated at $x \mapsto ix$, so $S(x) = e^{ix} - 1$ for all $x \in \mathbb{R}$ (convergence is a non-issue, since e^x converges for all $x \in (-\infty, \infty)$). Since $|S(x)^{12}| = 1$, $|S(x)| = 1$ as a result. If $S(x) + 1 = e^{ix} = a + bi$, then $S(x) = (a-1) + bi$ has magnitude 1, so a and b must satisfy the simultaneous equations $a^2 + b^2 = 1$ and $(a-1)^2 + b^2 = 1$, implying $a = \frac{1}{2}$ and $b = \pm\frac{\sqrt{3}}{2}$. For $b = \frac{\sqrt{3}}{2}$, we have $S(x) + 1 = \frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{\frac{\pi i}{3}}$, so $x = \frac{\pi}{3}$, and likewise $x = \frac{5\pi}{3}$ for $b = -\frac{\sqrt{3}}{2}$. Altogether, the product of the possible values of x in the interval $[0, 2\pi)$ is $\boxed{\frac{5\pi^2}{9}}$.

18. Equilateral triangle ABC has side length 2. When $\triangle ABC$ is rotated by $45°$ clockwise about vertex A to map vertices A, B, and C to A', B', and C', respectively, in triangle $A'B'C'$ (where $A' = A$), the area of overlap between triangles ABC and $A'B'C'$ can be written in the form $\frac{p+\sqrt{q}-\sqrt{r}-\sqrt{s}}{t}$, where p, q, r, s, and t are positive integers. Compute $p + q + r + s + t$.

Answer: $\boxed{44}$

WLOG assign coordinates $A = (0,0)$, $B = (-1, -\sqrt{3})$, $C = (1, -\sqrt{3})$. Treating the $45°$ rotation as a movement along the circle with center A and radius 2, one can consider B as positioned at an angle of $240°$ counter-clockwise relative to the positive x-axis, i.e. relative to $(2,0)$, and map it to $(2\cos(240°), 2\sin(240°)) = (-1, -\sqrt{3})$, so that $B' = (2\cos(195°), 2\sin(195°)) = \left(-\frac{\sqrt{6}+\sqrt{2}}{2}, \frac{\sqrt{2}-\sqrt{6}}{2}\right)$; likewise, $C = (2\cos(300°), 2\sin(300°))$, so that $C' = (2\cos(255°), 2\sin(255°)) = \left(\frac{\sqrt{2}-\sqrt{6}}{2}, \frac{-\sqrt{6}-\sqrt{2}}{2}\right)$.

At this point, we draw the following diagram:



The line segments $\overline{AB}$ and $\overline{B'C'}$ intersect where $y = x\sqrt{3} = -x - \sqrt{6}$, or at $\left(\frac{\sqrt{6}-3\sqrt{2}}{2}, \frac{3\sqrt{2}-3\sqrt{6}}{2}\right)$. The line segments $\overline{BC}$ and $\overline{B'C'}$ intersect where $y = -x - \sqrt{6} = -\sqrt{3} \implies x = \sqrt{3} - \sqrt{6}$, or at $(\sqrt{3} - \sqrt{6}, -\sqrt{3})$. Likewise, $\overline{BC}$ and $\overline{AC'}$ intersect where $y = -\sqrt{3}$ and $y = (2 + \sqrt{3})x$, or at $(3 - 2\sqrt{3}, -\sqrt{3})$.

Henceforth, we want to compute the area of the quadrilateral whose vertices are $(0, 0)$, $(3-2\sqrt{3}, -\sqrt{3})$, $(\sqrt{3}-\sqrt{6}, -\sqrt{3})$, and $\left(\frac{\sqrt{6}-3\sqrt{2}}{2}, \frac{3\sqrt{2}-3\sqrt{6}}{2}\right)$. From the diagram above, designating the intersection point of $\overline{AC'}$ with $\overline{BC}$ as D, the intersection point of $\overline{AB}$ with $\overline{B'C'}$ as E, and the intersection point of $\overline{AB}$ with $\overline{B'C'}$ as F, we want $[\triangle ABD] - [\triangle BFE]$. This is equal to

$$\frac{1}{2}(4-2\sqrt{3})(\sqrt{3}) - \frac{1}{2}(1+\sqrt{3}-\sqrt{6})\left(\frac{3\sqrt{6}-3\sqrt{2}-2}{2}\right) = (2\sqrt{3}-3) - \frac{(3\sqrt{6}-3\sqrt{2}-2)(1+\sqrt{3}-\sqrt{6})}{4}$$

$$= \frac{(8\sqrt{3}-12) - (-20+6\sqrt{2}+4\sqrt{3}+2\sqrt{6})}{4} = \frac{4-3\sqrt{2}+2\sqrt{3}-\sqrt{6}}{4} = \frac{4-\sqrt{18}+\sqrt{12}-\sqrt{6}}{4}.$$

Finally, $p + q + r + s + t = 4 + 18 + 12 + 6 + 4 = \boxed{44}$.

19. For how many ordered pairs (a, b) of positive integers with $1 \le a, b \le 10$ does the equation $4x^4 + 2ax^3 + bx^2 + ax + 1 = 0$ have exactly two real solutions in x (up to multiplicity)?

Answer: $\boxed{72}$

Dividing through by x^2 (since $x \neq 0$) and substituting $k := 2x + \frac{1}{x}$, we obtain $k^2 + ak + (b - 4) = 0$, so that

$$k = \frac{-a \pm \sqrt{a^2 - 4b + 16}}{2}.$$

For there to be two real solutions x, we must have either $a^2 - 4b > -16$, *and* either exactly one value of $k \in (-\infty, -2\sqrt{2}) \cup (2\sqrt{2}, \infty)$ (which is the range of $f(x) = 2x + \frac{1}{x}$, as can be seen by AM-GM), or $k = -2\sqrt{2}$ for the - sign and $k = 2\sqrt{2}$ for the + sign, which happens exactly when $a = 0$ and $b = -4$ (but we can safely ignore this case). Or we can have $a^2 - 4b = -16$ exactly, in which case we have a double root at $k = -\frac{a}{2}$. Since a is positive, this implies $a > 4\sqrt{2}$, but this would imply $4b > 48$, which contradicts our assumption that $b \le 10$.

In other words, either

$$-a + \sqrt{a^2 - 4b + 16} > 4\sqrt{2}$$

and

$$-4\sqrt{2} < -a - \sqrt{a^2 - 4b + 16} < 4\sqrt{2},$$

or

$$-4\sqrt{2} < -a + \sqrt{a^2 - 4b + 16} < 4\sqrt{2}$$

and

$$-a - \sqrt{a^2 - 4b + 16} < -4\sqrt{2}.$$

In the first case,

$$\begin{aligned} -a + \sqrt{a^2 - 4b + 16} &> 4\sqrt{2} \\ \implies a^2 - 4b + 16 &> (4\sqrt{2} + a)^2 = a^2 + 8\sqrt{2}a + 32 \\ \implies b &< -2\sqrt{2}a - 4, \end{aligned}$$

which is not possible.

In the second case,

$$-4\sqrt{2} < -a + \sqrt{a^2 - 4b + 16}$$

$$\implies a^2 - 4b + 16 > a^2 - 8\sqrt{2}a + 32$$
$$\implies -4b > -8\sqrt{2}a + 16$$
$$\implies b < 2\sqrt{2}a - 4,$$

and

$$-a + \sqrt{a^2 - 4b + 16} < 4\sqrt{2}$$
$$\implies a^2 - 4b + 16 < 32 + 8\sqrt{2}a + a^2$$
$$\implies -4b < 16 + 8\sqrt{2}a$$
$$\implies b > -4 - 2\sqrt{2}a$$

(trivially true), and also

$$-a - \sqrt{a^2 - 4b + 16} < -4\sqrt{2}$$
$$\implies a + \sqrt{a^2 - 4b + 16} > 4\sqrt{2}$$
$$\implies a^2 - 4b + 16 > 32 - 8\sqrt{2}a + a^2$$
$$\implies -4b > 16 - 8\sqrt{2}a$$
$$\implies b < 2\sqrt{2}a - 4,$$

so we get the same condition as before.

It now suffices to count ordered pairs (a, b) of positive integers with $1 \le a, b \le 10$ such that $b < 2\sqrt{2}a - 4$ and $a^2 - 4b \ge -16$. Note that the condition $a^2 - 4b \ge -16$ is actually redundant, since even in the worst (actually unachievable) case of $b = 2\sqrt{2}a - 4$, we get $a^2 - 4b = a^2 - 8\sqrt{2}a + 16$, and $a^2 - 8\sqrt{2}a + 16 \ge -16$ for all a, since $a^2 - 8\sqrt{2}a + 32 = (a - 4\sqrt{2})^2 \ge 0$ for all a. So we indeed can count those pairs (a, b) with $b < 2\sqrt{2}a - 4$. For $a = 1$, we get no such pairs. For $a = 2$, $b = 1$ only; for $a = 3$, $b \le 4$; for $a = 4$, $b \le 7$; for $a \ge 5$, $b \le 10$. It follows that the total number of ordered pairs (a, b) is $1 + 4 + 7 + 6 \cdot 10 = \boxed{72}$.

20. For each real number $x \neq 0$, define
$$f(x) := \frac{x+1}{x},$$
and define
$$g(x, h) := \frac{f(x+h) - f(x)}{h}.$$
For some integer $c \neq 0, -1$, there exist exactly two distinct real values of x such that $g(x, c+1) - g(x, c) = x$. Compute the sum of the squares of the possible values of c.

Answer: $\boxed{5}$

First note that $\frac{x+1}{x} = 1 + \frac{1}{x}$, so that, whenever two values of $f(x)$ are subtracted from each other, the 1s cancel out – this will simplify calculation steps to come. The equation $g(x, c+1) - g(x, c) = x$ simplifies to

$$\frac{f(x+c+1) - f(x)}{c+1} - \frac{f(x+c) - f(x)}{c} = x$$
$$\implies \frac{\frac{1}{x+c+1} - \frac{1}{x}}{c+1} - \frac{\frac{1}{x+c} - \frac{1}{x}}{c} = x.$$

Multiplying through by $c(c+1) = c^2 + c$ gives

$$c\left(\frac{1}{x+c+1} - \frac{1}{x}\right) - (c+1)\left(\frac{1}{x+c} - \frac{1}{x}\right) = (c^2 + c)x$$

$$\implies \frac{-c^2 - c}{x(x+c+1)} + \frac{c^2 + c}{x(x+c)} = (c^2 + c)x$$

$$\implies \frac{-c^2-c}{x+c+1}+\frac{c^2+c}{x+c}=x^2(c^2+c)$$
$$\implies \frac{c^2+c}{(x+c)(x+c+1)}=x^2(c^2+c)$$
$$\implies x^2((x+c)^2+(x+c))=1$$
$$\implies x^4+x^3(2c+1)+x^2(c^2+c)-1=0.$$

Observe that, if $c-1$ yields real solutions for x, so does $-c$, since $(c-1)^2+(c-1)=c^2-c$. We can test $c=-2$ and $c=1$ (by Descartes' law of signs) to obtain exactly two real solutions for x (where a solution to the first equation lies in $[0,1]$ by the intermediate value theorem, and thus two solutions exist by the conjugate pair theorem, but no others, as is easily checkable by similar methods; and likewise for the second equation), and we may find that $c \geq 2$ and $c \leq -3$ give four real solutions. Altogether, we get that $c \in \{-2, 1\}$, and the sum of the squares of the possible values of c is $\boxed{5}$.