pen Tournament

nline

Contest File

Invitiational Math Tournament (Middle School)

Saturday, April 2, 2022

Qualifying Round (Middle School)

1. How many dots are in a formation with 1000 rows, where rows 1 and 1000 consist of a single dot, row 2 and 999 consist of 3 dots, rows 3 and 998 consists of 5 dots, and rows 4-997 each consist of 7 dots?

2. What is the sum of all positive integers x satisfying $|x(9-x)| < 72$?

3. How many positive integers have digits summing to 5 and no digits of zero?

4. Four fair six-sided dice, each labeled with the positive integers from 1 through 6 inclusive, are rolled. What is the probability that no two distinct dice come up with numbers whose product is a perfect square? Express your answer as a common fraction.

5. For how many positive integers $n \leq 100$ is $n(n+1)^2(n+2)$ divisible by 88?

6. Regular hexagon $ABCDEF$ has side length 3. Points G, H, I, J, K, and L lie on $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, and $\overline{FA}$ respectively such that $AG = BH = CI = DJ = EK = FL = 1$. Compute the area of hexagon $GHIJKL$. Express your answer as a common fraction in simplest radical form.

7. Label the positions of the letters in the word CYBERMATH with the positive integers from 1 to 9, inclusive, from left to right. Let S be the sum of the products of the left-to-right positions of A, B, and C, respectively, over all 9! permutations of CYBERMATH. Compute S mod 10000.

8. Triangle ABC has the property that the length of $\overline{AC}$ is an integer. Point D lies on $\overline{AC}$ with $m\angle ABD = m\angle DBC$. If the area of $\triangle ABD$ is 15 and the area of $\triangle BCD$ is 30, compute the smallest possible length of $\overline{AC}$.

9. A positive integer n is called *cube-special* if there exist distinct positive integers a and b with $a^3 - na = b^3 - nb$. How many positive integers less than or equal to 100 are cube-special?

10. How many permutations of $(1, 2, 3, 4, 5, 6, 7, 8)$ do not have any two consecutive positive integers in increasing order and in adjacent positions? For example, $(2, 1, 4, 3)$ is one such permutation of $(1, 2, 3, 4)$, but $(3, 1, 2, 4)$ is not.

Live Round (Middle School)

1. Let rectangle $ABCD$ have $AB = 2$. Point $E \neq C, D$ lies on $\overline{CD}$ such that $\triangle AED$ is similar to $\triangle ABE$. Show that E must be the midpoint of $\overline{CD}$.

2. Let $f(x) = ax^3 + bx^2 + cx + d$ be a quadratic polynomial with positive integer coefficients. Prove that the sum of the squares of the roots of $f(x)$ does not depend on d. If this sum is 1, what is the smallest possible value of $f(1)$?

3. Show that, for each positive integer $1 \leq z \leq 9$, the numbers of 10-digit positive integers whose digits sum to 10 with exactly z zeros and $9 - z$ zeros are equal.