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Contest Solutions

Invitational Math Tournament (Middle School)

Saturday, April 2, 2022

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Qualifying Round

1. How many dots are in a formation with 1000 rows, where rows 1 and 1000 consist of a single dot, row 2 and 999 consist of 3 dots, rows 3 and 998 consists of 5 dots, and rows 4-997 each consist of 7 dots?

   Answer: $\boxed{6976}$

   This is a $7 \cdot 1000$ array with triangular formations of $1 + 2 + 3 = 6$ dots removed at each of the four corners, hence consisting of $7000 - 24 = \boxed{6976}$ dots.

2. What is the sum of all positive integers x satisfying $|x(9 - x)| < 72$?

   Answer: $\boxed{105}$

   From $-72 < x(9 - x) < 72$, we get $x \leq 14$, so the requested sum is $\frac{14 \cdot 15}{2} = \boxed{105}$.

3. How many positive integers have digits summing to 5 and no digits of zero?

   Answer: $\boxed{16}$

   We can inductively show that there are 2^{s-1} such numbers for a sum of $s \leq 9$, hence $\boxed{16}$ numbers with $s = 5$. (This is because we can place a 1 either before the string of sum $s - 1$, or after it.) It is also not difficult to count the 16 possibilities directly.

4. Four fair six-sided dice, each labeled with the positive integers from 1 through 6 inclusive, are rolled. What is the probability that no two distinct dice come up with numbers whose product is a perfect square? Express your answer as a common fraction.

   Answer: $\boxed{\frac{1}{6}}$

   If any two dice come up with the same number, those dice will multiply to a perfect square. Thus, we certainly must have all four dice rolls be distinct (but this is not in itself a sufficient condition). Note that $(1, 4)$ is also a bad pair, but there are no others where the rolls are different. Excluding the $\binom{4}{2} = 6$ choices that have both 1 and 4 from the $\binom{6}{4} = 15$ choices of four distinct dice rolls, we get 9 good roll choices and their 4! permutations. Altogether, the probability of rolling a good 4-tuple is $\frac{9 \cdot 4!}{6^4} = \boxed{\frac{1}{6}}$.

5. For how many positive integers $n \leq 100$ is $n(n + 1)^2(n + 2)$ divisible by 88?

   Answer: $\boxed{20}$

   One of n, $n + 1$, or $n + 2$ must be a multiple of 11, so $n \equiv 0, 9, 10 \bmod 11$. In addition, we must have three factors of 2; if n is even, then n and $n + 2$ are even while $n + 1$ is odd, so one of them must be a multiple of 4 (but this is guaranteed, so all even n congruent to $0, 9, 10 \bmod 11$ work). If n is odd, on the other hand, $n + 1$ is the only even number among n, $n + 1$, and $n + 2$, so $n + 1$ must be a multiple of 4 (in order for $(n + 1)^2$ to be a multiple of 8 as needed). This implies $n \equiv 3 \bmod 4$ in the odd case. Altogether, in the case where n is even and congruent to $0, 9, 10 \bmod 11$, we get 3 solutions in every block of 22 by the Chinese remainder theorem, hence 12 solutions up to $n = 88$, and also the solution $n = 98$, for 13 solutions. In the second case, where $n \equiv 3 \bmod 4$ and also congruent to $0, 9, 10 \bmod 11$, we get 3 solutions in every block of 44 by CRT, hence 6 solutions up to and including $n = 888$. We have $n = 99$ as a solution as well, so there are a total of $13 + 7 = \boxed{20}$ possible values for n.

6. Regular hexagon $ABCDEF$ has side length 3. Points G, H, I, J, K, and L lie on $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, and $\overline{FA}$ respectively such that $AG = BH = CI = DJ = EK = FL = 1$. Compute the area of

hexagon $GHIJKL$. Express your answer as a common fraction in simplest radical form.

Answer: $\boxed{\dfrac{21\sqrt{3}}{2}}$

The desired area is the area of $ABCDEF$, which is $\frac{27\sqrt{3}}{2}$, less six times the area of a triangle with side lengths 2 and 1 adjacent to a vertex angle of 120°. By the area formula $\frac{ab\sin C}{2}$, the area of each such triangle is $\frac{\sqrt{3}}{2}$, so the area of $GHIJKL$ is $\boxed{\dfrac{21\sqrt{3}}{2}}$.

7. Label the positions of the letters in the word CYBERMATH with the positive integers from 1 to 9, inclusive, from left to right. Let S be the sum of the products of the left-to-right positions of A, B, and C, respectively, over all 9! permutations of CYBERMATH. Compute S mod 10000.

Answer: $\boxed{4000}$

Note that S will be $6! \cdot 3! = 4320$ times the sum over all $\binom{9}{3}$ *unique* choices of the positions of A, B, and C, without respect to order. This sum is equal to $1(2 \cdot 42 + 3 \cdot 39 + 4 \cdot 35 + 5 \cdot 30 + 6 \cdot 24 + 7 \cdot 17 + 8 \cdot 9) + 2(3 \cdot 39 + 4 \cdot 35 + 5 \cdot 30 + 6 \cdot 24 + 7 \cdot 17 + 8 \cdot 9) + 3(4 \cdot 35 + \cdots + 8 \cdot 9) + \cdots + 7 \cdot 8 \cdot 9$, where we continually drop the first term of the sum; through some calculation, we get that this is $826+1484+1875+1940+1675+1146+504 = 9450$. Multiplying by 4320 gives $S \equiv \boxed{4000}$ mod 10000.

8. Triangle ABC has the property that the length of $\overline{AC}$ is an integer. Point D lies on $\overline{AC}$ with $m\angle ABD = m\angle DBC$. If the area of $\triangle ABD$ is 15 and the area of $\triangle BCD$ is 30, compute the smallest possible length of $\overline{AC}$.

Answer: $\boxed{12}$

By the angle bisector theorem, we know that $\frac{AB}{BC} = \frac{15}{30} = \frac{1}{2}$, so we can let $AB = x$ and $BC = 2x$. Since the area of $\triangle ABC$ is 45, the length of the altitude from A to foot $F \in \overline{BC}$ of the perpendicular to $\overline{BC}$ is $\frac{45}{x}$. Thus, $BF = \sqrt{x^2 - \frac{2025}{x^2}}$, and so $FC = 2x - \sqrt{x^2 - \frac{2025}{x^2}}$. Then $AC =$

$$\sqrt{\frac{2025}{x^2} + \left(2x - \sqrt{x^2 - \frac{2025}{x^2}}\right)^2} = \sqrt{\frac{2025}{x^2} + 4x^2 - 4\sqrt{x^4 - 2025} + \left(x^2 - \frac{2025}{x^2}\right)} = \sqrt{5x^2 - 4\sqrt{x^4 - 2025}}.$$

Therefore, we want to minimize $5x^2 - 4\sqrt{x^4 - 2025}$, while also making sure it is a perfect square. Set it equal to m, and we want the smallest possible value of m; this yields $m - 5x^2 = 4\sqrt{x^4 - 2025}$, or $16x^4 - 180^2 = m^2 - 10mx^2 + 25x^4 \implies 9x^2 - 10mx^2 + (m^2 - 180^2) = 0$. For m to be a minimum, we want the discriminant of this quadratic to be zero, so that $100m^2 = 36(m^2 - 180^2) \implies m = 135$. Therefore, $AC \geq \sqrt{135}$, and the smallest possible integer value of AC is then $\boxed{12}$.

9. A positive integer n is called *cube-special* if there exist distinct positive integers a and b with $a^3 - na = b^3 - nb$. How many positive integers less than or equal to 100 are cube-special?

Answer: $\boxed{22}$

We have $a^3 - b^3 = na - nb = n(a-b)$; since $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$, we have $n = a^2 + ab + b^2$. For $b - a = 1$, we get $n = a^2 + a(a+1) + (a+1)^2 = 3a^2 + 3a + 1$, so $n \in \{7, 19, 37, 61, 91\}$ in order to be less than or equal to 100 and cube-special in this case. For $b - a = 2$, we similarly get $n = 3a^2 + 6a + 4$, so that $n \in \{13, 28, 49, 76\}$. For $b - a = 3$, $n = 3a^2 + 9a + 9$, so $n \in \{21, 39, 63, 93\}$. For $b - a = 4$, $n \in \{31, 52, 79\}$; for $b - a = 5$, $n \in \{43, 67, 97\}$; for $b - a = 6$, $n \in \{57, 84\}$; for $b - a = 7$, $n \in \{73\}$; and for $b - a = 8$, $n \in \{91\}$. For $k := b - a \geq 9$, we'd have $n = 3a^2 + 3ka + k^2$, which for $a = 1$ evaluates to $3 + 3k + k^2$, but this is more than 100 for $k \geq 9$. Thus, $\{7, 13, 19, 21, 28, 31, 37, 39, 43, 49, 52, 57, 61, 63, 67, 73, 76, 79, 84, 91, 93, 97\}$ is the set of all cube-special numbers up to 100, of which there are $\boxed{22}$.

10. How many permutations of $(1, 2, 3, 4, 5, 6, 7, 8)$ do not have any two consecutive positive integers in increasing order and in adjacent positions? For example, $(2, 1, 4, 3)$ is one such permutation of $(1, 2, 3, 4)$, but $(3, 1, 2, 4)$ is not.

    Answer: $\boxed{16687}$

    Let $P(n)$ be this number of permutations of $(1, 2, 3, \cdots, n)$. We define $P(n)$ recursively, with base cases $P(1) = P(2) = 1$. For $n \geq 3$, we may form a desired permutation of $P(n)$ either by placing the number n between two consecutive numbers in a permutation of length $n - 2$ with *exactly* one pair of consecutive numbers (which can be done in $n - 2$ ways), or in $n - 1$ possible positions (i.e. not immediately to the right of $n - 1$) given a desired permutation of length $n - 1$. This gives rise to the recurrence $P(n) = (n - 1)P(n - 1) + (n - 2)P(n - 2)$, from which $P(8) = \boxed{16687}$.

A note on Live Round scoring

Each of the 3 problems is scored out of 10 points, for a maximum of 30 points. The following is a rough guideline for the assignment of scores:

- 10 points: Perfect solution.
- 9 points: Extremely minor computational error (sign error, addition error, etc).
- 8 points: Mostly correct, but with a few minor computational errors or a minor mis-application of a formula or idea.
- 7 points: Has the structure of a correct proof, but slightly sloppy or imprecise (although not incorrect) in the execution.
- 6 points: Has the general structure of a correct proof, but the execution is slightly sloppy and handwavy; in addition, there may be a few missing key components.
- 5 points: Half-complete; usually one part of a problem is done correctly but not another, or the student has forgotten a critical component of the proof (a good example is showing minimality/maximality but not achievability).
- 4 points: Possible misapplication of a critical idea, but on the right track.
- 3 points: A considerable amount of nontrivial progress that has the potential to lead to a solution with significant work.
- 2 points: Some nontrivial progress.
- 1 point: Some tangential observations related to the problem.
- 0 points: No or only entirely trivial progress. **An answer (even if correct) with no justification should be scored zero.**

There is some built-in leeway here, and scores are ultimately assigned at each judge's personal discretion.

Live Round

1. Let rectangle $ABCD$ have $AB = 2$. Point $E \neq C, D$ lies on $\overline{CD}$ such that $\triangle AED$ is similar to $\triangle ABE$. Show that E must be the midpoint of $\overline{CD}$.

   Let $BC = u$ and $ED = x$; then from $\frac{AE}{AB} = \frac{ED}{BE} = \frac{DA}{EA}$, we get that $\frac{\sqrt{x^2+u^2}}{2} = \frac{x}{\sqrt{(2-x)^2+u^2}} = \frac{u}{\sqrt{x^2+u^2}}$. From $\frac{\sqrt{x^2+u^2}}{2} = \frac{u}{\sqrt{x^2+u^2}}$, it follows that $x^2+u^2 = 2u$, or $x = \sqrt{2u - u^2}$. Then $\frac{\sqrt{2u}}{2} = \frac{\sqrt{2u-u^2}}{\sqrt{(2-\sqrt{2u-u^2})^2+u^2}}$, so we get that $\frac{u}{2} = \frac{2u-u^2}{4-4\sqrt{2u-u^2}+2u} \implies 2-2\sqrt{2u-u^2}+u = 2-u \implies u = \sqrt{2u-u^2} \implies u = 0, 1$. Since $E \neq D$, $ED = u \neq 0$, so $u = 1$ and E is the midpoint of $\overline{CD}$.

Scoring guidelines:

- A valiant attempt, but one that does not employ the correct ideas of similarity, should receive either 0/10 or 1/10 points, depending on whether the attempt contains other ideas that are potentially applicable to the problem (at judge's discretion).
- If the student correctly applies similarity ratios: minimum of 2/10 points.
- Reaching $x = \sqrt{2u-u^2}$ is worth 5/10 points.
- Solving the equations correctly is worth 9/10 points *if the explanation is entirely thorough and clear.*
- Unclear explanation of equation solving: deduct points as appropriate, perhaps 2-3 points from the total score. (There is some leeway with this item.)
- Forgetting about $u \neq 0$ is a 1-point deduction.
- Other silly mistake (e.g. a^2+b^2 instead of $\sqrt{a^2+b^2}$), with everything else correct and clearly explained: deduct 1 point.

2. Let $f(x) = ax^3+bx^2+cx+d$ be a quadratic polynomial with positive integer coefficients. Prove that the sum of the squares of the roots of $f(x)$ does not depend on d. If this sum is 1, what is the smallest possible value of $f(1)$?

Answer: $\boxed{9}$

Let the roots be r, s, and t; then $r^2+s^2+t^2 = (r+s+t)^2 - 2(rs+rt+st) = \left(\frac{b}{a}\right)^2 - 2\frac{c}{a}$ by Vieta's formulas, which is independent of d. For $\frac{b^2}{a^2} - \frac{2c}{a} = \frac{b^2-2ac}{a^2} = 1$, we have $b^2-2ac = a^2$. If $a = 1$, we get $b^2 - 2c = 1$, implying that b is odd, so $b \geq 3$. Then $c = 4$, and $f(1) = a+b+c+d \geq 9$ (since $d \geq 1$, being a positive integer). On the other hand, if $a \geq 2$ and is even, from $b^2 = a^2+2ac$, we'd get that b is even, hence $b \geq 4$ (since $b = 2$ implies $a^2+2ac = 4$, which is impossible with $a = 2$). So then $a^2+2ac \geq 16$; with $a = 2$, this gives $c \geq 3$, or $a+b+c+d \geq 10$, and with $a \geq 4$, this gives $a+b+c+d \geq 4+4+1+1 = 10$. With $a \geq 3$ odd, $b \geq 4$, since $3^2+2\cdot3\cdot1 = 15$. Thus, $a+b+c+d \geq 9$, and 9 is indeed minimal.

Scoring guidelines:

- A valiant, but handwavy and non-rigorous attempt with an *incorrect answer* should receive either 0/10 or 1/10 points, depending on whether the attempt contains ideas that are potentially applicable to the problem (at judge's discretion).
- A valiant attempt with a correct answer, but a very handwavy and non-rigorous proof, should receive between 1/10 and 3/10 points (at the judge's discretion).
- Writing the correct factorization of $r^2+s^2+t^2$ is worth a minimum of 2/10 points.
- Correctly applying Vieta's formulas to complete the first part of the problem is worth 4/10 points.
- Given the previous item: a general casework approach on a, with not much success, is worth 5/10 points.
- Identifying $f(1) = a+b+c+d$ is worth +1 point on its own.
- Using a mod 4 argument in the casework is worth a minimum of 3 points (on top of the correct use of Vieta's formulas, this is a minimum of 7/10 points).
- Identifying 9 as achievable, but not attempting a minimality argument, is worth at most 6/10 points.
- Silly mistake, with everything else correct and clearly explained: deduct 1 point.

3. Show that, for each positive integer $1 \leq z \leq 8$, the numbers of 10-digit positive integers whose digits sum to 10 with exactly z zeros and $9 - z$ zeros are equal.

   The z zeros can go in 9 places (anything but the leftmost digit), and $\binom{9}{z} = \binom{9}{9-z}$. We have $10 - z$ nonzero digits summing to 10 in the case that we have z zeros, and $1 + z$ nonzero digits summing to 10 in the case that we have $9 - z$ zeros, or $10 - z$ digits summing to z as opposed to $1 + z$ digits summing to $9 - z$. By stars-and-bars, we have $\binom{((10-z)+z-1)}{((10-z)-1)} = \binom{9}{9-z}$ ways in the first case, and $\binom{((1+z)+(9-z)-1)}{((1+z)-1)} = \binom{9}{z}$ ways in the second case; these are the same for all $1 \leq z \leq 8$.

   Scoring guidelines:

   - A valiant, but handwavy and non-rigorous attempt, should receive at most 2/10 points, depending on the extent to which the attempt contains ideas that are potentially applicable to the problem (at judge's discretion).
   - Observing that we have equal numbers of ways to place the zeros is worth a minimum of 2/10 points.
   - General understanding of/a sketch of how stars-and-bars might be broadly applicable is worth 4-5 points, at the judge's discretion.
   - Rigorously applying stars-and-bars is worth 10/10 points, before deductions.
   - Silly mistake, with everything else correct and clearly explained: deduct 1 point.